

APPROXIMATION ORDERS OF REAL NUMBERS BY β -EXPANSIONS

LULU FANG, MIN WU AND BING LI*

ABSTRACT. We prove that almost all real numbers (with respect to Lebesgue measure) are approximated by the convergents of their β -expansions with the exponential order β^{-n} . Moreover, the Hausdorff dimensions of sets of the real numbers which are approximated by all other orders, are determined. These results are also applied to investigate the orbits of real numbers under β -transformation, the shrinking target type problem, the Diophantine approximation and the run-length function of β -expansions.

1. INTRODUCTION

Let $\beta > 1$ be a real number and $T_\beta : [0, 1] \rightarrow [0, 1]$ be the β -transformation defined as

$$T_\beta(x) = \beta x - [\beta x],$$

where $[x]$ denotes the greatest integer not exceeding x . Then every $x \in [0, 1]$ has an β -expansion, namely

$$x = \frac{\varepsilon_1(x, \beta)}{\beta} + \frac{\varepsilon_2(x, \beta)}{\beta^2} + \cdots + \frac{\varepsilon_n(x, \beta) + T_\beta^n}{\beta^n} = \sum_{n=1}^{\infty} \frac{\varepsilon_n(x, \beta)}{\beta^n}, \quad (1.1)$$

where $\varepsilon_1(x, \beta) = [\beta x]$ and $\varepsilon_{n+1}(x, \beta) = \varepsilon_1(T_\beta^n x, \beta)$ are called the *digits* of the β -expansion of x ($n \in \mathbb{N}$). Sometimes we write the sequence $\varepsilon(x, \beta) = (\varepsilon_1(x, \beta), \varepsilon_2(x, \beta), \dots, \varepsilon_n(x, \beta), \dots)$ as the β -expansion of x . If there exists some $n_0 \in \mathbb{N}$ such that $\varepsilon_n(x, \beta) = 0$ for all $n \geq n_0$, we say that the β -expansion of x is *finite*. Otherwise, it is said to be *infinite*. Such an expansion was first introduced by Rényi [34] who proved that there exists a unique T_β -invariant measure μ_β equivalent to the Lebesgue measure when β is not an integer; while it had been well known that the Lebesgue measure is T_β -invariant when β is an integer. Furthermore, Gel'fond [18] and Parry [30] independently found the density formula for this invariant measure with respect to (w.r.t.) the Lebesgue measure. Philipp [32] showed that the dynamical system $([0, 1], \mathcal{B}, T_\beta, \mu_\beta)$ is an exponentially mixing measure-preserving system, where \mathcal{B} is the Borel σ -algebra on $[0, 1]$. Later, Hofbauer [20] proved that μ_β is the unique measure of maximal entropy for T_β . Aaronson and Nakada [1] obtained the β -transformation T_β is exponential φ -mixing. And they also showed that T_β is exponential ψ -mixing if and only if $\inf_{n \geq 1} T_\beta^n 1 > 0$ (see Bradley [6] for the definitions of φ -mixing and ψ -mixing). Some arithmetic, metric and fractal properties of β -expansions were studied extensively in the literature, such as [2, 5, 12, 17, 23, 27, 29, 31, 35, 36, 39] and the references therein.

For any real number $x \in [0, 1]$, we denote the partial sums of the form (1.1) by

$$\omega_n(x, \beta) = \frac{\varepsilon_1(x, \beta)}{\beta} + \frac{\varepsilon_2(x, \beta)}{\beta^2} + \cdots + \frac{\varepsilon_n(x, \beta)}{\beta^n}$$

and call them the *convergents* of the β -expansion of x ($n \in \mathbb{N}$). In the following, we write $\varepsilon_n(x)$ and $\omega_n(x)$ instead of $\varepsilon_n(x, \beta)$ and $\omega_n(x, \beta)$ respectively if there is no confusion. Since $T_\beta^n x \in [0, 1]$, by the representation (1.1), we have that the sequence $\{\omega_n(x) : n \geq 1\}$ converges to x as n tends to infinity for any $x \in [0, 1]$. A nature question is how fast $\omega_n(x)$ converges to x . The following theorem gives it a quantitative answer. Let λ denote the Lebesgue measure on $[0, 1]$.

Theorem 1.1. *Let $\beta > 1$ be a real number. Then for λ -almost all $x \in [0, 1]$,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log_\beta(x - \omega_n(x)) = -1.$$

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* Corresponding author.

Roughly speaking, Theorem 1.1 means that $x - \omega_n(x) \approx \beta^{-n}$ for λ -almost all $x \in [0, 1)$. That is to say, x is approximated by its convergents $\omega_n(x)$ with exponential order β^{-n} for λ -almost all $x \in [0, 1)$. A further question is whether there exist some points with other approximation order than β^{-n} . If yes, how large is the set of such points. More precisely, we would like to know how many real numbers can be approximated with other orders $\beta^{-\phi(n)}$, where ϕ is a positive function defined on \mathbb{N} . In other words, we are interested in the following set

$$\left\{ x \in [0, 1) : \lim_{n \rightarrow \infty} \frac{1}{\phi(n)} \log_{\beta}(x - \omega_n(x)) = -1 \right\}. \quad (1.2)$$

The problem on the approximation orders is a longstanding topic in mathematics, for example, the approximation of functions or the numbers. The approximation problems on the representations of real numbers have been widely investigated, see [11, 21, 33] for continued fractions, see [13, 14] for Oppenheim expansions, see [3, 9] for Lüroth expansions.

Replacing the limit of the quantity $\frac{1}{\phi(n)} \log_{\beta}(x - \omega_n(x))$ in (1.2) with limsup, we obtain

Proposition 1.2. *Let $\beta > 1$ be a real number and ϕ be a positive function defined on \mathbb{N} . Define*

$$A_{\phi} := \left\{ x \in [0, 1) : \limsup_{n \rightarrow \infty} \frac{1}{\phi(n)} \log_{\beta}(x - \omega_n(x)) = -1 \right\}.$$

Then

- (i) *If $\liminf_{n \rightarrow \infty} \phi(n)/n > 1$, then A_{ϕ} is countable at most.*
- (ii) *If $\limsup_{n \rightarrow \infty} \phi(n)/n < 1$, then A_{ϕ} is empty.*

We use the notation \dim_{H} to denote the Hausdorff dimension (see Falconer [10]) and let $1/\infty := 0$ with the convention. Replacing the limit with liminf in the set (1.2), we have the following.

Theorem 1.3. *Let $\beta > 1$ be a real number and ϕ be a positive function defined on \mathbb{N} . Define $\eta := \liminf_{n \rightarrow \infty} \phi(n)/n$ and*

$$B_{\phi} := \left\{ x \in [0, 1) : \liminf_{n \rightarrow \infty} \frac{1}{\phi(n)} \log_{\beta}(x - \omega_n(x)) = -1 \right\}.$$

Then

- (i) *If $0 \leq \eta < 1$, then B_{ϕ} is empty.*
- (ii) *If additionally assume ϕ is nondecreasing and $\eta \geq 1$, then $\dim_{\text{H}} B_{\phi} = 1/\eta$.*

Taking $\phi(n) = \alpha n$, Theorem 1.3 gives the Hausdorff dimensions of the following level sets, which shows that these level sets have a rich multifractal structure.

Corollary 1.4. *Let $\beta > 1$ be a real number. Then the set*

$$\left\{ x \in [0, 1) : \liminf_{n \rightarrow \infty} \frac{1}{n} \log_{\beta}(x - \omega_n(x)) = -\alpha \right\}$$

has Hausdorff dimension α^{-1} for any $\alpha \geq 1$; otherwise it is empty.

Application of Corollary 1.4 implies the set of points such that Theorem 1.1 does not hold, has full Hausdorff dimension.

Corollary 1.5. *Let $\beta > 1$ be a real number. Then*

$$\dim_{\text{H}} \left\{ x \in [0, 1) : \lim_{n \rightarrow \infty} \frac{1}{n} \log_{\beta}(x - \omega_n(x)) \neq -1 \right\} = 1.$$

2. PRELIMINARIES

2.1. Basic definitions and properties for β -expansions.

Definition 2.1. *An n -block $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ is said to be admissible for β -expansions if there exists $x \in [0, 1)$ such that $\varepsilon_i(x) = \varepsilon_i$ for all $1 \leq i \leq n$. An infinite sequence $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n, \dots)$ is admissible if $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ is admissible for all $n \geq 1$.*

We denote by Σ_β^n the collection of all admissible sequences of length n ($n \in \mathbb{N}$) and by Σ_β that of all infinite admissible sequences. The following result of Rényi [34] implies that the dynamical system $([0,1], T_\beta)$ admits $\log \beta$ as its topological entropy.

Proposition 2.2 ([34]). *Let $\beta > 1$ be a real number. For any $n \geq 1$,*

$$\beta^n \leq \#\Sigma_\beta^n \leq \beta^{n+1}/(\beta - 1),$$

where $\#$ denotes the number of elements of a finite set.

Definition 2.3. *Let $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in \Sigma_\beta^n$. We define*

$$I(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) = \{x \in [0, 1) : \varepsilon_i(x) = \varepsilon_i \text{ for all } 1 \leq i \leq n\}$$

and call it the n -th cylinder of β -expansion. Furthermore, if $T_\beta^n I(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) = [0, 1)$, we say $I(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ is full.

Remark 2.4. *The n -th order cylinder $I(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ is a left-closed and right-open interval with left endpoint*

$$\frac{\varepsilon_1}{\beta} + \frac{\varepsilon_2}{\beta^2} + \dots + \frac{\varepsilon_n}{\beta^n}.$$

Moreover, the length of $I(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ satisfies $|I(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)| \leq 1/\beta^n$. We stress that there is no nontrivial lower bound for the length of a n -th cylinder, which can be much smaller than β^{-n} . However, it is clear that $I(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ is full if and only if $|I(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)| = 1/\beta^n$. The properties of full cylinders were investigated by Fan and Wang [12]. In [7], the authors gave a full characterization of full cylinders and studied the distribution of full cylinders in the unit interval.

Denote

$$\mathcal{A} = \begin{cases} \{0, 1, \dots, \beta - 1\}, & \text{if } \beta \text{ is an integer;} \\ \{0, 1, \dots, [\beta]\}, & \text{if } \beta \text{ is not an integer.} \end{cases}$$

Let $(\mathcal{A}^\mathbb{N}, \sigma)$ be the symbolic dynamics with the shift transformation σ on $\mathcal{A}^\mathbb{N}$. The finite word ε^n and the infinite sequence ε^∞ means $\underbrace{\varepsilon\varepsilon\dots\varepsilon}_n$ and $\varepsilon\varepsilon\dots\varepsilon\dots$ respectively for a finite word $\varepsilon \in \mathcal{A}^\mathbb{N}$ ($n \in \mathbb{N}$).

When β is an integer, Σ_β is simply $\mathcal{A}^\mathbb{N}$ (or more precisely $\mathcal{A}^\mathbb{N} = \mathcal{S}_\beta$ defined below); when β is not an integer, Σ_β was characterized by Parry [30] using the infinite β -expansion of the number 1, denoted by $\varepsilon^*(1, \beta)$, which can be obtained in the following: if the β -expansion of the number 1 is finite, i.e., $\varepsilon(1, \beta) = (\varepsilon_1(1), \varepsilon_2(1), \dots, \varepsilon_n(1), 0^\infty)$ with $\varepsilon_n(1) \neq 0$ for some $n \geq 1$, we define by $\varepsilon^*(1, \beta)$ the infinite β -expansion of the number 1 as $(\varepsilon_1(1), \varepsilon_2(1), \dots, \varepsilon_{n-1}(1), \varepsilon_n(1) - 1)^\infty$; if the β -expansion of the number 1 is infinite, we keep it and still write $\varepsilon^*(1, \beta)$ instead of $\varepsilon(1, \beta)$ in this case. To state the following proposition, we give two notations \prec and \preceq , the lexicographical orders on $\mathcal{A}^\mathbb{N}$. That is, let $\varepsilon = \varepsilon_1\varepsilon_2\dots\varepsilon_n\dots$ and $\varepsilon' = \varepsilon'_1\varepsilon'_2\dots\varepsilon'_n\dots$ both belong to $\mathcal{A}^\mathbb{N}$, then $\varepsilon \prec \varepsilon'$ means that there exists $k \geq 1$ such that $\varepsilon_i = \varepsilon'_i$ for all $1 \leq i < k$ and $\varepsilon_k < \varepsilon'_k$, and $\varepsilon \preceq \varepsilon'$ means that $\varepsilon \prec \varepsilon'$ or $\varepsilon = \varepsilon'$.

Proposition 2.5. ([30, Theorem 3]) *Let $\beta > 1$ be a real number and $\varepsilon^*(1, \beta)$ be the infinite β -expansion of the number 1.*

(i) $\omega \in \Sigma_\beta$ if and only if

$$\sigma^n(\omega) \prec \varepsilon^*(1, \beta) \text{ for all } n \geq 0.$$

(ii) *The function $\beta \mapsto \varepsilon^*(1, \beta)$ is increasing w.r.t. β . Therefore, if $1 < \beta_1 < \beta_2$, then*

$$\Sigma_{\beta_1} \subset \Sigma_{\beta_2}, \quad \Sigma_{\beta_1}^n \subset \Sigma_{\beta_2}^n \text{ for all } n \geq 1.$$

Let \mathcal{S}_β be the closure of the set Σ_β . It is clear to see $\mathcal{S}_\beta = \mathcal{A}^\mathbb{N}$ when β is an integer and otherwise, $(\mathcal{S}_\beta, \sigma|_{\mathcal{S}_\beta})$ is a subshift of $(\mathcal{A}^\mathbb{N}, \sigma)$, where $\sigma|_{\mathcal{S}_\beta}$ is the restriction of σ to \mathcal{S}_β . Proposition 2.5 implies the following characterization of \mathcal{S}_β .

Corollary 2.6 ([30]). *Let $\beta > 1$ be a real number and $\varepsilon^*(1, \beta)$ be the infinite β -expansion of the number 1. Then*

$$\mathcal{S}_\beta = \{\omega \in \mathcal{A}^\mathbb{N} : \sigma^n(\omega) \preceq \varepsilon^*(1, \beta) \text{ for all } n \geq 0\}.$$

In 1989, Blanchard [5] outlined a classification for all numbers $\beta > 1$ according to the topological properties of \mathcal{S}_β . Later, the Lebesgue measures and Hausdorff dimensions of all classes were calculated by Schmeling [35]. Denote by $\ell_n(1, \beta)$ the length of the longest strings of zeroes just after the n -th digit in the β -expansion of the number 1, namely,

$$\ell_n(1, \beta) = \sup \{k \geq 0 : \varepsilon_{n+1}^*(1) = \cdots = \varepsilon_{n+k}^*(1) = 0\}.$$

Recently, Li and Wu [27] provided another classification of $\beta > 1$ by the growth of $\ell_n(1, \beta)$ as follows:

$$A_0 = \left\{ \beta > 1 : \limsup_{n \rightarrow \infty} \ell_n(1, \beta) < +\infty, \text{ i.e., } \{\ell_n(1, \beta)\}_{n \geq 1} \text{ is bounded} \right\}$$

and $A_1 = (1, +\infty) \setminus A_0$. Then A_0 is dense in $(1, +\infty)$ (see Parry [30]). It is worth pointing out that all β 's such that \mathcal{S}_β is a subshift of finite type are contained in A_0 and $\beta \in A_0$ if and only if \mathcal{S}_β satisfies the specification property. Buzzi [8] proved that the set of $\beta > 1$ such that the β -transformation T_β has the specification property is of zero Lebesgue measure. Furthermore, Schmeling [35] proved that A_0 has full Hausdorff dimension and Li et al. [25] showed that A_1 is of full Lebesgue measure.

The following lemma from [27] gives a way to get full cylinders.

Lemma 2.7. ([27]) *Let $\beta > 1$ be a real number and $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in \Sigma_\beta^n$. Denote $M_n(\beta) = \max_{1 \leq k \leq n} \{\ell_k(1, \beta)\}$, then for any $m > M_n(\beta)$, the cylinder*

$$I(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n, \underbrace{0, \dots, 0}_m)$$

is a full cylinder and its length equals $\beta^{-(n+m)}$.

The following result characterizes the sizes of cylinders by the classification in A_0 .

Proposition 2.8. ([27]) $\beta \in A_0$ if and only if there exists a positive constant C_0 such that for all $x \in [0, 1)$ and $n \geq 1$,

$$C_0 \beta^{-n} \leq |I(\varepsilon_1(x), \varepsilon_2(x), \dots, \varepsilon_n(x))| \leq \beta^{-n}.$$

2.2. Approximation method for the β -shift. Define a projection function $\pi_\beta : \mathcal{S}_\beta \rightarrow [0, 1]$ as following

$$\pi_\beta(\omega) = \sum_{i=1}^{\infty} \frac{\omega_i}{\beta^i}$$

for any $\omega = (\omega_1, \omega_2, \dots, \omega_n, \dots) \in \mathcal{S}_\beta$. Then π_β is one-to-one except at the countable many points for which their β -expansions are finite and the restriction of π_β to which is two-to-one. Let $\beta > \beta' > 1$. Since $\Sigma_{\beta'} \subset \Sigma_\beta$, we know that

$$H_\beta^{\beta'} := \pi_\beta(\Sigma_{\beta'}) = \left\{ \sum_{i=1}^{\infty} \frac{\omega'_i}{\beta^i} : (\omega'_1, \dots, \omega'_n, \dots) \in \Sigma_{\beta'} \right\}$$

is a Cantor set of $\pi_\beta(\Sigma_\beta) = [0, 1)$. Define the function $h : H_\beta^{\beta'} \rightarrow [0, 1)$ as

$$h(x) = \pi_{\beta'}(\varepsilon(x, \beta))$$

for any $x \in H_\beta^{\beta'}$.

Proposition 2.9. ([4]) *Let $\beta > \beta' > 1$.*

(i) *For any $x \in H_\beta^{\beta'}$, we have $\varepsilon(h(x), \beta') = \varepsilon(x, \beta)$.*

(ii) *The function h is bijective and strictly increasing on $H_\beta^{\beta'}$.*

(iii) *The function h is continuous on $H_\beta^{\beta'}$.*

(iv) *If additionally assume that $\beta' \in A_0$ with $M = \sup\{\ell_n(1, \beta') : n \geq 1\}$, then h is Hölder continuous on $H_\beta^{\beta'}$. More precisely,*

$$|h(x) - h(y)| \leq \beta'^{M+2} |x - y|^{\frac{\log \beta'}{\log \beta}}$$

for any $x, y \in H_\beta^{\beta'}$.

The function h induces a method to provide a lower bound of the Hausdorff dimension of a given set $E \subset [0, 1)$. Firstly, consider a subset $E \cap H_\beta^{\beta'}$ of E and use the Hölder function h in Proposition 2.9 to transfer it to $h(E \cap H_\beta^{\beta'})$, whose Hausdorff dimension may be easier to be obtained by choosing $\beta' \in A_0$ or β' satisfying that $\mathcal{S}_{\beta'}$ is subshift of finite type. Secondly, give a lower bound of the Hausdorff dimension of $h(E \cap H_\beta^{\beta'})$ and then by the Hölder exponent of h in Proposition 2.9 (iv) have a lower bound of the Hausdorff dimension of $E \cap H_\beta^{\beta'}$, also that of E . That is,

$$\dim_H E \geq \dim_H(E \cap H_\beta^{\beta'}) \geq \frac{\log \beta'}{\log \beta} \dim_H h(E \cap H_\beta^{\beta'}).$$

Finally, let β' approximates to β . In Section 4, we will apply this approximation method to prove our desired results.

3. METRIC RESULTS

For any real number $x \in [0, 1)$ and $n \geq 1$, it is clear to see that

$$\frac{1}{\beta^{n+\ell_n(x)+1}} \leq x - \omega_n(x) \leq \frac{1}{\beta^{n+\ell_n(x)}} \quad (3.1)$$

where $\ell_n(x) = \sup \{k \geq 0 : \varepsilon_{n+1}(x) = \cdots = \varepsilon_{n+k}(x) = 0\}$ is the length of the longest string of zeros just after the n -th digit in the β -expansion of x . The quantity $\ell_n(x)$ has been used by Fang, Wu and Li [15, 16]. The inequalities (3.1) indicate that $\ell_n(x)$ plays an important role in the approximation theory of β -expansions.

To estimate $\ell_n(x)$, we first define $r_n(x)$ the maximal length of the strings of zero's in the block of the first n digits of the β -expansion of $x \in [0, 1)$. That is,

$$r_n(x) = \sup \{k \geq 0 : \varepsilon_{i+1}(x) = \cdots = \varepsilon_{i+k}(x) = 0 \text{ for some } 0 \leq i \leq n - k\}.$$

Sometimes the quantity $r_n(x)$ is called *run-length function* for β -expansions. Tong et al. [40] studied the order of magnitude of $r_n(x)$ and investigated the Hausdorff dimensions for the corresponding exceptional sets.

Lemma 3.1. ([40, Theorem 1.1]) *Let $\beta > 1$ be a real number. Then for λ -almost all $x \in [0, 1)$,*

$$\lim_{n \rightarrow \infty} \frac{r_n(x)}{\log_\beta n} = 1.$$

Combing this with the relation between $r_n(x)$ and $\ell_n(x)$, we have the following

Proposition 3.2. *Let $\beta > 1$ be a real number. Then for λ -almost all $x \in [0, 1)$,*

$$\limsup_{n \rightarrow \infty} \frac{\ell_n(x)}{\log_\beta n} = 1.$$

Moreover, for any real number $x \in [0, 1)$ whose β -expansion is infinite, we have

$$\liminf_{n \rightarrow \infty} \frac{\ell_n(x)}{\log_\beta n} = 0.$$

Proof. We first prove the result of \liminf part. Let $x \in [0, 1)$ be a real number whose β -expansion is infinite. Then there exists a subsequence of digits $\{\varepsilon_{n_k}(x) : k \geq 1\}$ with $\varepsilon_{n_k}(x) \neq 0$ for all $k \geq 1$. So, by the definition of $\ell_n(x)$, we have $\ell_{n_k-1}(x) = 0$ for any $k \geq 1$ and hence

$$\liminf_{n \rightarrow \infty} \frac{\ell_n(x)}{\log_\beta n} \leq \liminf_{k \rightarrow \infty} \frac{\ell_{n_k-1}(x)}{\log_\beta(n_k-1)} = 0.$$

Therefore, $\liminf_{n \rightarrow \infty} \ell_n(x)/(\log_\beta n) = 0$ by the definition of $\ell_n(x)$. Now we turn to the result for \limsup part. Let B be the set such that the Lemma 3.1 does not hold and let $A = [0, 1) \setminus B$. Then $\lambda(A) = 1$. For any $x \in A$ and $n \geq 1$, we know that $r_{n+\ell_n(x)}(x) = \max_{1 \leq k \leq n} \ell_k(x)$ by the definitions of $\ell_n(x)$

and $r_n(x)$. So there exists $1 \leq k_n := k_n(x) \leq n$ such that $\ell_{k_n}(x) = \max_{1 \leq k \leq n} \ell_k(x)$. Therefore, $r_{n+\ell_n(x)}(x) = \ell_{k_n}(x)$ and hence

$$\frac{\ell_{k_n}(x)}{\log_\beta k_n} \geq \frac{r_{n+\ell_n(x)}(x)}{\log_\beta n} \geq \frac{r_{n+\ell_n(x)}(x)}{\log_\beta(n + \ell_n(x))}.$$

Combining this with Lemma 3.1, we deduce that

$$\limsup_{n \rightarrow \infty} \frac{\ell_n(x)}{\log_\beta n} \geq \limsup_{n \rightarrow \infty} \frac{\ell_{k_n}(x)}{\log_\beta k_n} \geq \liminf_{n \rightarrow \infty} \frac{r_{n+\ell_n(x)}(x)}{\log_\beta(n + \ell_n(x))} \geq \liminf_{n \rightarrow \infty} \frac{r_n(x)}{\log_\beta n} = 1.$$

On the other hand, note that $r_{n+\ell_n(x)}(x) = \max_{1 \leq k \leq n} \ell_k(x) \geq \ell_n(x)$, so

$$\limsup_{n \rightarrow \infty} \frac{\ell_n(x)}{\log_\beta n} \leq \limsup_{n \rightarrow \infty} \frac{r_{n+\ell_n(x)}(x)}{\log_\beta(n + \ell_n(x))} \leq \limsup_{n \rightarrow \infty} \frac{r_n(x)}{\log_\beta n} = 1.$$

Therefore, we get that $\limsup_{n \rightarrow \infty} \ell_n(x)/(\log_\beta n) = 1$ for λ -almost all $x \in [0, 1)$. \square

Now we are ready to prove Theorem 1.1 and Proposition 1.2.

Proof of Theorem 1.1. For any $x \in [0, 1)$ and $n \geq 1$, by (3.1), we obtain that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log_\beta(x - \omega_n(x)) = - \lim_{n \rightarrow \infty} \frac{\ell_n(x)}{n} - 1.$$

In view of Proposition 3.2, we know that $\lim_{n \rightarrow \infty} \ell_n(x)/n = 0$ for λ -almost all $x \in [0, 1)$. Thus,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log_\beta(x - \omega_n(x)) = -1$$

for λ -almost all $x \in [0, 1)$. \square

Proof of Proposition 1.2. (i): Notice that $\phi(n) \rightarrow \infty$ as $n \rightarrow \infty$ since $\liminf_{n \rightarrow \infty} \phi(n)/n > 1$. So, it follows from the inequalities (3.1) that

$$\limsup_{n \rightarrow \infty} \frac{1}{\phi(n)} \log_\beta(x - \omega_n(x)) = -1 \quad \text{if and only if} \quad \liminf_{n \rightarrow \infty} \frac{n + \ell_n(x)}{\phi(n)} = 1$$

for any $x \in [0, 1)$. We claim that

$$\left\{ x \in [0, 1) : \liminf_{n \rightarrow \infty} \frac{n + \ell_n(x)}{\phi(n)} = 1 \right\} \subseteq \{ x \in [0, 1) : \text{the } \beta\text{-expansion of } x \text{ is finite} \}.$$

In fact, suppose that $x \in [0, 1)$ whose β -expansion is infinite, that is, there exists a subsequence $\{n_k\}_{k \geq 1}$ such that $\varepsilon_{n_k-1}(x) \neq 0$. Then $\ell_{n_k}(x) = 0$ by the definition of $\ell_n(x)$ and hence

$$\liminf_{n \rightarrow \infty} \frac{n + \ell_n(x)}{\phi(n)} \leq \liminf_{k \rightarrow \infty} \frac{n_k}{\phi(n_k)} = \frac{1}{\limsup_{k \rightarrow \infty} \phi(n_k)/n_k} \leq \frac{1}{\liminf_{k \rightarrow \infty} \phi(n_k)/n_k}.$$

Therefore,

$$\liminf_{n \rightarrow \infty} \frac{n + \ell_n(x)}{\phi(n)} \leq \frac{1}{\liminf_{n \rightarrow \infty} \phi(n)/n} < 1.$$

Thus we get the desired result since the set of the points with finite β -expansions is a countable set.

(ii): Since $\limsup_{n \rightarrow \infty} \phi(n)/n < 1$, we deduce that

$$\liminf_{n \rightarrow \infty} \frac{n + \ell_n(x)}{\phi(n)} \geq \liminf_{n \rightarrow \infty} \frac{n}{\phi(n)} = \frac{1}{\limsup_{n \rightarrow \infty} \phi(n)/n} > 1.$$

By (3.1), we know that

$$\frac{\log_\beta(x - \omega_n(x))}{\phi(n)} \leq -\frac{n + \ell_n(x)}{\phi(n)},$$

which implies that

$$\limsup_{n \rightarrow \infty} \frac{1}{\phi(n)} \log_\beta(x - \omega_n(x)) < -1.$$

□

4. DIMENSIONAL RESULTS

The inequalities (3.1) say that

$$\left\{x \in [0, 1) : \liminf_{n \rightarrow \infty} \frac{1}{\phi(n)} \log_{\beta}(x - \omega_n(x)) = -1\right\} = \left\{x \in [0, 1) : \limsup_{n \rightarrow \infty} \frac{n + \ell_n(x)}{\phi(n)} = 1\right\}$$

if $\phi(n) \rightarrow \infty$ as $n \rightarrow \infty$. Now it leads us to consider the Hausdorff dimension of the set

$$F_{\phi} = \left\{x \in [0, 1) : \limsup_{n \rightarrow \infty} \frac{\ell_n(x)}{\phi(n)} = 1\right\}.$$

Firstly, we will give an upper bound for the Hausdorff dimension of the set F_{ϕ} .

4.1. Upper bound. Denote

$$E_{\phi} = \{x \in [0, 1) : \ell_n(x) \geq \phi(n) \text{ i.o.}\},$$

where *i.o.* means infinitely often. It is clear that $F_{\phi} \subseteq E_{(1-\delta)\phi}$ for any $0 < \delta < 1$. So we will determine the upper bound of the Hausdorff dimension of the set F_{ϕ} by giving an upper bound of the Hausdorff dimension of E_{ϕ} .

Lemma 4.1. *Let $\beta > 1$ be a real number. Assume that ϕ is a nonnegative function defined on \mathbb{N} . Then*

$$\dim_{\text{H}} E_{\phi} \leq \frac{1}{1 + \liminf_{n \rightarrow \infty} \phi(n)/n}.$$

Proof. The upper bound can be obtained by considering the natural covering system. Notice that

$$E_{\phi} = \{x \in [0, 1) : \ell_n(x) \geq \phi(n) \text{ i.o.}\} = \bigcap_{N=1} \bigcup_{n=N} \{x \in [0, 1) : \ell_n(x) \geq \phi(n)\}$$

and

$$\{x \in [0, 1) : \ell_n(x) \geq \phi(n)\} \subseteq \bigcup_{(\varepsilon_1, \dots, \varepsilon_n) \in \Sigma_{\beta}^n} I(\varepsilon_1, \dots, \varepsilon_n, \underbrace{0, \dots, 0}_{\lfloor \phi(n) \rfloor}),$$

so, for any $N \geq 1$, we obtain that

$$E_{\phi} \subseteq \bigcup_{n=N} \bigcup_{(\varepsilon_1, \dots, \varepsilon_n) \in \Sigma_{\beta}^n} I(\varepsilon_1, \dots, \varepsilon_n, \underbrace{0, \dots, 0}_{\lfloor \phi(n) \rfloor}).$$

Let $s = 1/(1 + \liminf_{n \rightarrow \infty} \phi(n)/n)$. Then $0 \leq s \leq 1$. For any $\delta > 0$ and $N \geq 1$, by the definition of Hausdorff measure, we have that

$$\begin{aligned} \mathcal{H}^{s+\delta}(E_{\phi}) &\leq \sum_{n=N}^{\infty} \sum_{(\varepsilon_1, \dots, \varepsilon_n) \in \Sigma_{\beta}^n} |I(\varepsilon_1, \dots, \varepsilon_n, \underbrace{0, \dots, 0}_{\lfloor \phi(n) \rfloor})|^{s+\delta} \\ &\leq \sum_{n=N}^{\infty} \frac{\beta^{n+1}}{\beta-1} \cdot \left(\frac{1}{\beta^{n+\lfloor \phi(n) \rfloor}}\right)^{s+\delta} = \frac{\beta}{\beta-1} \sum_{n=N}^{\infty} \left(\frac{1}{\beta^n}\right)^{t_n}, \end{aligned} \quad (4.1)$$

where $t_n := (1 + \lfloor \phi(n) \rfloor/n)(s + \delta) - 1$ and the second inequality follows from Proposition 2.2 and the fact $|I(\varepsilon_1, \dots, \varepsilon_n)| \leq 1/\beta^n$ for any $(\varepsilon_1, \dots, \varepsilon_n) \in \Sigma_{\beta}^n$.

When $s = 0$, i.e., $\liminf_{n \rightarrow \infty} \phi(n)/n = +\infty$. So, $\liminf_{n \rightarrow \infty} \lfloor \phi(n) \rfloor/n \geq \delta^{-1}$. So we know that

$$\liminf_{n \rightarrow \infty} t_n = \liminf_{n \rightarrow \infty} \left(1 + \frac{\lfloor \phi(n) \rfloor}{n}\right) \delta - 1 \geq (1 + \delta^{-1})\delta - 1 = \delta.$$

When $0 < s \leq 1$, we deduce that

$$\liminf_{n \rightarrow \infty} t_n = \liminf_{n \rightarrow \infty} \left(1 + \frac{\lfloor \phi(n) \rfloor}{n}\right) (s + \delta) - 1 = (s + \delta)/s - 1 = \delta/s \geq \delta.$$

In conclusion, we have that $\liminf_{n \rightarrow \infty} t_n \geq \delta$. So, there exists $N_\delta > 0$ such that $t_n \geq \delta/2$ for all $n \geq N_\delta$. Applying $N = N_\delta$ to (4.1), we obtain that

$$\mathcal{H}^{s+\delta}(E_\phi) \leq \frac{\beta}{\beta-1} \sum_{n=N_\delta}^{\infty} \left(\frac{1}{\beta^n}\right)^{t_n} \leq \frac{\beta}{\beta-1} \sum_{n=N_\delta}^{\infty} \frac{1}{\beta^{n\delta/2}} < +\infty.$$

Therefore, $\dim_{\text{H}} E_\phi \leq s + \delta$. By the arbitrariness of $\delta > 0$, we have $\dim_{\text{H}} E_\phi \leq s$. \square

Note that $F_\phi \subset E_{(1-\delta)\phi}$ for any $0 < \delta < 1$, by Lemma 4.1, we know that

$$\dim_{\text{H}} F_\phi \leq \dim_{\text{H}} E_{(1-\delta)\phi} \leq \frac{1}{1 + (1-\delta) \liminf_{n \rightarrow \infty} \phi(n)/n}.$$

Let $\delta \rightarrow 0^+$, we deduce that

$$\dim_{\text{H}} F_\phi \leq \frac{1}{1 + \liminf_{n \rightarrow \infty} \phi(n)/n}.$$

4.2. Lower bound. In this subsection, the main aim is to determine a lower bound for the Hausdorff dimension of the set F_ϕ . First, we give a lower bound of the Hausdorff dimension of F_ϕ for $\beta \in A_0$, where A_0 is as defined in Section 2.1. Then go on to estimate the lower bound of the Hausdorff dimension of F_ϕ for any $\beta > 1$ using the approximation method stated in Section 2.2.

The lower bound of the Hausdorff dimension of F_ϕ is yielded by constructing a Cantor-like subset of F_ϕ . The mass distribution principle (see [10, Proposition 4.2]) is a classical tool to give a lower bound estimation for the Hausdorff dimension of a set. In the classical form of the mass distribution principle, we need to estimate the measure of an arbitrary ball, while the following modified mass distribution principle tells us that, for β -expansions, it is sufficient to consider only the measure of cylinders.

Proposition 4.2. ([7, Proposition 1.3]) *Let E be a Borel measurable set in $[0, 1)$ and ν be a Borel measure with $\nu(E) > 0$. Assume that there exist a positive constant c and an integer n_0 such that*

$$\nu(I_n) \leq c|I_n|^s$$

for any $n \geq n_0$ and any n -th order cylinder I_n . Then $\dim_{\text{H}} E \geq s$.

4.2.1. The cases of bases in A_0 .

Lemma 4.3. *Let $\beta \in A_0$. Assume that ϕ is a positive and nondecreasing function defined on \mathbb{N} with $\phi(n) \rightarrow \infty$ as $n \rightarrow \infty$. Then*

$$\dim_{\text{H}} F_\phi \geq \frac{1}{1 + \liminf_{n \rightarrow \infty} \phi(n)/n}. \quad (4.2)$$

The following is devoted to proving this lemma in this subsection.

Let $0 \leq s = 1/(1 + \liminf_{n \rightarrow \infty} \phi(n)/n) \leq 1$. When $s = 0$, (4.2) holds trivially. Now let $0 < s \leq 1$. Denote $M = \sup_{n \geq 1} \ell_n(1, \beta)$ and $m = M + 1$, then we have that $0 < M, m < +\infty$ since $\beta \in A_0$. Let $\{n_i\}_{i \geq 1}$ be a sequence with

$$\lim_{i \rightarrow \infty} \frac{\phi(n_i)}{n_i} = \liminf_{n \rightarrow \infty} \frac{\phi(n)}{n}. \quad (4.3)$$

For any $0 < \delta < s$, we choose a largely sparse subsequence $\{n_{i_j}\}_{j \geq 1}$ of $\{n_i\}_{i \geq 1}$ (for simplicity, we still denote by $\{n_i\}_{i \geq 1}$) such that

$$\frac{1}{[\phi(n_1)]} < \frac{\delta}{4(m+1 - \log_\beta(\beta-1))}, \quad n_1 > 1, \quad n_i > n_{i-1} + 1 + [\phi(n_{i-1})]$$

and

$$\frac{\delta}{2}(n_i + [\phi(n_i)]) \geq \sum_{j=1}^{i-1} ([\phi(n_j)] + r_j + 1) + (m+1 - \log_\beta(\beta-1)) \sum_{j=1}^{i-1} (k_j + 1), \quad (4.4)$$

where $k_j = \left\lfloor \frac{n_{j+1} - n_j - 1}{[\phi(n_j)]} \right\rfloor \geq 1$ is an integer and $r_j = n_{j+1} - n_j - 1 - k_j[\phi(n_j)]$ is the remainder. We sometimes write the right-hand side formula of (4.4) as U_i (with the convention $U_1 := 0$). The

first condition $\lfloor \phi(n_1) \rfloor^{-1} < \frac{\delta}{4(m+1-\log_\beta(\beta-1))}$ assures that the sequence satisfying (4.4) can be chosen. Moreover, it also guarantees that

$$\frac{1}{\lfloor \phi(n_i) \rfloor} < \frac{1 + \delta - s}{m + 1 - \log_\beta(\beta - 1)}$$

for $0 < \delta < s$. That is, $W_i := \lfloor \phi(n_i) \rfloor(s - \delta - 1) + m + 1 - \log_\beta(\beta - 1) < 0$. The following proof will be divided into three steps to make use of Proposition 4.2.

Step 1 Construction of a Cantor-like subset. We denote the two subsets of integers

$$\mathcal{I}_1 = \{n_i + j : i \geq 1, 1 \leq j \leq \lfloor \phi(n_i) \rfloor\} \cup \{n_i + j \lfloor \phi(n_i) \rfloor - m + r : i \geq 1, 2 \leq j \leq k_i, 1 \leq r \leq m\} \cup \Gamma$$

and

$$\mathcal{I}_2 = \{n_i, n_i + j \lfloor \phi(n_i) \rfloor + 1 : i \geq 1, 1 \leq j \leq k_i - 1\} \setminus \{n_1\} \cup \Lambda,$$

where $\Gamma = \cup_{i \geq 1} \Gamma_i$, $\Lambda = \cup_{i \geq 1} \Lambda_i$ and when $r_i \leq m$, $\Gamma_i = \{n_i + k_i \lfloor \phi(n_i) \rfloor + 1, \dots, n_{i+1} - 1\}$ and Λ_i is empty; if $r_i > m$, $\Gamma_i = \{n_{i+1} - m, \dots, n_{i+1} - 1\}$ and $\Lambda_i = \{n_i + k_i \lfloor \phi(n_i) \rfloor + 1\}$. Let \mathcal{D}_n be the n -th order cylinders $I(\varepsilon_1, \dots, \varepsilon_n)$ satisfying that $\varepsilon_k = 0$ if $k \in I_1$, $\varepsilon_k \neq 0$ if $k \in I_2$, and $\varepsilon_k \in \mathcal{A}$ if $k \notin \mathcal{I}_1 \cup \mathcal{I}_2$. Put

$$F = \bigcap_{n=1}^{\infty} \bigcup_{I(\varepsilon_1, \dots, \varepsilon_n) \in \mathcal{D}_n} I(\varepsilon_1, \dots, \varepsilon_n).$$

By the constructions of \mathcal{D}_n and F , and the monotonic property of ϕ , we claim that

$$F \subset F_\phi = \left\{ x \in [0, 1) : \limsup_{n \rightarrow \infty} \frac{\ell_n(x)}{\phi(n)} = 1 \right\}.$$

In fact, for any $n \geq 1$, there exists $k \in \mathbb{N}$ such that $n_k \leq n < n_{k+1}$. For any $x \in F$, by the construction of F and the monotonic property of ϕ , we know that

$$\ell_n(x) = \max\{\lfloor \phi(n_k) \rfloor, \lfloor \phi(n_{k-1}) \rfloor\} \leq \lfloor \phi(n_k) \rfloor \leq \phi(n)$$

and hence that

$$\limsup_{n \rightarrow \infty} \frac{\ell_n(x)}{\phi(n)} \leq 1.$$

Note that $\ell_{n_k}(x) = \lfloor \phi(n_k) \rfloor$, so $\lim_{k \rightarrow \infty} \ell_{n_k}(x)/\phi(n_k) = 1$. Therefore, $\limsup_{n \rightarrow \infty} \ell_n(x)/\phi(n) = 1$.

Step 2 Supporting measure. We now distribute a probability measure ν supported on F . We first give the definition of ν on cylinders. Notice that $n_1 > 1$, then $1 \notin \mathcal{I}_1 \cup \mathcal{I}_2$. For any $I(\varepsilon_1) \in \mathcal{D}_1$, we define $\nu(I(\varepsilon_1)) = |I(\varepsilon_1)|$ and $\nu(I(\varepsilon_1)) = 0$ if $I(\varepsilon_1) \notin \mathcal{D}_1$. Suppose that $\nu(I(\varepsilon_1, \dots, \varepsilon_n))$ is well defined for any $I(\varepsilon_1, \dots, \varepsilon_n) \in \mathcal{D}_n$, we define $\nu(I(\varepsilon_1, \dots, \varepsilon_n, \varepsilon_{n+1}))$ as follows:

$$\nu(I(\varepsilon_1, \dots, \varepsilon_n, \varepsilon_{n+1})) = \begin{cases} \nu(I(\varepsilon_1, \dots, \varepsilon_n)) & \text{if } n+1 \in \mathcal{I}_1; \\ \frac{|I(\varepsilon_1, \dots, \varepsilon_n, \varepsilon_{n+1})|}{|I(\varepsilon_1, \dots, \varepsilon_n) \setminus I(\varepsilon_1, \dots, \varepsilon_n, 0)|} \cdot \nu(I(\varepsilon_1, \dots, \varepsilon_n)) & \text{if } n+1 \in \mathcal{I}_2; \\ \frac{|I(\varepsilon_1, \dots, \varepsilon_n, \varepsilon_{n+1})|}{|I(\varepsilon_1, \dots, \varepsilon_n)|} \cdot \nu(I(\varepsilon_1, \dots, \varepsilon_n)) & \text{if } n+1 \notin \mathcal{I}_1 \cup \mathcal{I}_2, \end{cases}$$

and $\nu(I(\varepsilon_1, \dots, \varepsilon_n, \varepsilon_{n+1})) = 0$ if $I(\varepsilon_1, \dots, \varepsilon_n, \varepsilon_{n+1}) \notin \mathcal{D}_{n+1}$. Thus, the measure ν is well-defined on all cylinders since we can verify that

$$\sum_{(\varepsilon_1, \dots, \varepsilon_n) \in \Sigma_\beta^n} \nu(I(\varepsilon_1, \dots, \varepsilon_n)) = 1$$

and

$$\sum_{(\varepsilon_1, \dots, \varepsilon_n, \varepsilon_{n+1}) \in \Sigma_\beta^{n+1}} \nu(I(\varepsilon_1, \dots, \varepsilon_n, \varepsilon_{n+1})) = \nu(I(\varepsilon_1, \dots, \varepsilon_n)).$$

Notice that the set of all cylinders forms a semi-algebra, by Kolmogorov's extension theorem, ν can be extensively defined on the measurable space $([0, 1), \mathcal{B})$. Now we list some facts about the expression for the measure of a cylinder.

(I) When $n_i < n \leq n_i + \lfloor \phi(n_i) \rfloor$ ($i \geq 1$), we have

$$\nu(I(\varepsilon_1, \dots, \varepsilon_n)) = \nu(I(\varepsilon_1, \dots, \varepsilon_{n_i})).$$

(II) When $n_i + j\lfloor\phi(n_i)\rfloor < n \leq n_i + j\lfloor\phi(n_i)\rfloor + \lfloor\phi(n_i)\rfloor - m$ ($i \geq 1$ and $1 \leq j \leq k_i - 1$), we deduce that

$$\nu(I(\varepsilon_1, \dots, \varepsilon_n)) = \frac{|I(\varepsilon_1, \dots, \varepsilon_n)|}{\beta^{-(n_i+j\lfloor\phi(n_i)\rfloor)} - \beta^{-(n_i+j\lfloor\phi(n_i)\rfloor+1)}} \cdot \nu(I(\varepsilon_1, \dots, \varepsilon_{n_i+(j-1)\lfloor\phi(n_i)\rfloor+\lfloor\phi(n_i)\rfloor-m})).$$

In fact, the construction of \mathcal{D}_n and the distribution of the measure ν yield that

$$\nu(I(\varepsilon_1, \dots, \varepsilon_n)) = \frac{|I(\varepsilon_1, \dots, \varepsilon_n)|}{|I(\varepsilon_1, \dots, \varepsilon_{n-1})|} \cdot \nu(I(\varepsilon_1, \dots, \varepsilon_{n-1})). \quad (4.5)$$

Note that (4.5) is also true if we replace n by $(n-1)$, therefore, we obtain

$$\nu(I(\varepsilon_1, \dots, \varepsilon_n)) = \frac{|I(\varepsilon_1, \dots, \varepsilon_n)|}{|I(\varepsilon_1, \dots, \varepsilon_{n-1})|} \cdot \frac{|I(\varepsilon_1, \dots, \varepsilon_{n-1})|}{|I(\varepsilon_1, \dots, \varepsilon_{n-2})|} \cdot \nu(I(\varepsilon_1, \dots, \varepsilon_{n-2})).$$

Repeating the above procedure, we finally have

$$\nu(I(\varepsilon_1, \dots, \varepsilon_n)) = \frac{|I(\varepsilon_1, \dots, \varepsilon_n)|}{|I(\varepsilon_1, \dots, \varepsilon_{n_i+j\lfloor\phi(n_i)\rfloor+1})|} \cdot \nu(I(\varepsilon_1, \dots, \varepsilon_{n_i+j\lfloor\phi(n_i)\rfloor+1})). \quad (4.6)$$

It follows from $\varepsilon_{n_i+j\lfloor\phi(n_i)\rfloor+1} \neq 0$ and the distribution of the measure ν that

$$\begin{aligned} \nu(I(\varepsilon_1, \dots, \varepsilon_{n_i+j\lfloor\phi(n_i)\rfloor+1})) &= \frac{|I(\varepsilon_1, \dots, \varepsilon_{n_i+j\lfloor\phi(n_i)\rfloor+1})|}{|I(\varepsilon_1, \dots, \varepsilon_{n_i+j\lfloor\phi(n_i)\rfloor}) \setminus I(\varepsilon_1, \dots, \varepsilon_{n_i+j\lfloor\phi(n_i)\rfloor}, 0)|} \\ &\quad \times \nu(I(\varepsilon_1, \dots, \varepsilon_{n_i+j\lfloor\phi(n_i)\rfloor})) \end{aligned} \quad (4.7)$$

Since $\varepsilon_{n_i+(j-1)\lfloor\phi(n_i)\rfloor+\lfloor\phi(n_i)\rfloor-m+1} = \dots = \varepsilon_{n_i+j\lfloor\phi(n_i)\rfloor} = 0$, by the construction of \mathcal{D}_n and the distribution of the measure ν , we obtain that

$$\nu(I(\varepsilon_1, \dots, \varepsilon_{n_i+j\lfloor\phi(n_i)\rfloor})) = \nu(I(\varepsilon_1, \dots, \varepsilon_{n_i+(j-1)\lfloor\phi(n_i)\rfloor+\lfloor\phi(n_i)\rfloor-m})). \quad (4.8)$$

In view of Lemma 2.7 and the construction of \mathcal{D}_n , we know that the two cylinders $I(\varepsilon_1, \dots, \varepsilon_{n_i+j\lfloor\phi(n_i)\rfloor})$ and $I(\varepsilon_1, \dots, \varepsilon_{n_i+j\lfloor\phi(n_i)\rfloor}, 0)$ are both full. Hence

$$|I(\varepsilon_1, \dots, \varepsilon_{n_i+j\lfloor\phi(n_i)\rfloor}) \setminus I(\varepsilon_1, \dots, \varepsilon_{n_i+j\lfloor\phi(n_i)\rfloor}, 0)| = \beta^{-(n_i+j\lfloor\phi(n_i)\rfloor)} - \beta^{-(n_i+j\lfloor\phi(n_i)\rfloor+1)}.$$

Combining this with (4.6), (4.7) and (4.8), we get the desired result.

(III) When $n_i + j\lfloor\phi(n_i)\rfloor + \lfloor\phi(n_i)\rfloor - m < n \leq n_i + (j+1)\lfloor\phi(n_i)\rfloor$ ($i \geq 1$ and $1 \leq j \leq k_i - 1$), it is easy to check

$$\nu(I(\varepsilon_1, \dots, \varepsilon_n)) = \nu(I(\varepsilon_1, \dots, \varepsilon_{n_i+j\lfloor\phi(n_i)\rfloor+\lfloor\phi(n_i)\rfloor-m})).$$

(IV) When $n_i + k_i\lfloor\phi(n_i)\rfloor < n \leq n_{i+1}$ ($i \geq 1$). In this case, we should distinguish two cases according to the relationship between the remainder r_i and m .

(i) If $r_i \leq m$, by the construction of \mathcal{D}_n and the distribution of the measure ν , we know that

$$\nu(I(\varepsilon_1, \dots, \varepsilon_n)) = \nu(I(\varepsilon_1, \dots, \varepsilon_{n_i+(k_i-1)\lfloor\phi(n_i)\rfloor+\lfloor\phi(n_i)\rfloor-m})).$$

(ii) If $r_i > m$, we need to distinguish two cases according to the position of n .

(1) For $n_i + k_i\lfloor\phi(n_i)\rfloor < n \leq n_{i+1} - m - 1$, being similar to (II), we obtain

$$\nu(I(\varepsilon_1, \dots, \varepsilon_n)) = \frac{|I(\varepsilon_1, \dots, \varepsilon_n)|}{\beta^{-(n_i+k_i\lfloor\phi(n_i)\rfloor)} - \beta^{-(n_i+k_i\lfloor\phi(n_i)\rfloor+1)}} \cdot \nu(I(\varepsilon_1, \dots, \varepsilon_{n_i+(k_i-1)\lfloor\phi(n_i)\rfloor+\lfloor\phi(n_i)\rfloor-m})).$$

(2) For $n_{i+1} - m - 1 < n \leq n_{i+1}$, we get that

$$\nu(I(\varepsilon_1, \dots, \varepsilon_n)) = \nu(I(\varepsilon_1, \dots, \varepsilon_{n_{i+1}-m-1})).$$

Step 3 Estimation on the ν -measure of cylinders. We claim that

$$\nu(I(\varepsilon_1, \dots, \varepsilon_{n_2})) \leq \frac{\beta^{(m+1-\log_\beta(\beta-1))(k_1+1)+\lfloor\phi(n_1)\rfloor+r_1+1}}{\beta^{n_2-n_1}} \cdot \nu(I(\varepsilon_1, \dots, \varepsilon_{n_1})).$$

We distinguish two cases to prove this statement according to the relation between r_1 and m .

- $r_1 \leq m$

By the construction of \mathcal{D}_n , we know $\varepsilon_{n_1+k_1\lfloor\phi(n_1)\rfloor-m+1} = \dots = \varepsilon_{n_2-1} = 0$ and $\varepsilon_{n_2} \neq 0$. Since $|I(\varepsilon_1, \dots, \varepsilon_{n_2})| \leq \beta^{-n_2}$, being similar to the Case (II) in Step 2, we obtain

$$\begin{aligned} \nu(I(\varepsilon_1, \dots, \varepsilon_{n_2})) &= \frac{|I(\varepsilon_1, \dots, \varepsilon_{n_2})|}{\beta^{n_2-1} - \beta^{n_2}} \cdot \nu(I(\varepsilon_1, \dots, \varepsilon_{n_1+(k_1-1)\lfloor\phi(n_1)\rfloor+\lfloor\phi(n_1)\rfloor-m})) \\ &\leq \frac{1}{\beta-1} \cdot \nu(I(\varepsilon_1, \dots, \varepsilon_{n_1+(k_1-1)\lfloor\phi(n_1)\rfloor+\lfloor\phi(n_1)\rfloor-m})). \end{aligned} \quad (4.9)$$

Now it remains to estimate the measure $\nu(I(\varepsilon_1, \dots, \varepsilon_{n_1+(k_1-1)\lfloor\phi(n_1)\rfloor+\lfloor\phi(n_1)\rfloor-m}))$. Being similar to the Case (II) in Step 2, we deduce that

$$\begin{aligned} &\nu(I(\varepsilon_1, \dots, \varepsilon_{n_1+(k_1-1)\lfloor\phi(n_1)\rfloor+\lfloor\phi(n_1)\rfloor-m})) \\ &= \frac{|I(\varepsilon_1, \dots, \varepsilon_{n_1+(k_1-1)\lfloor\phi(n_1)\rfloor+\lfloor\phi(n_1)\rfloor-m})|}{\beta^{-(n_1+(k_1-1)\lfloor\phi(n_1)\rfloor)} - \beta^{-(n_1+(k_1-1)\lfloor\phi(n_1)\rfloor+1)}} \cdot \nu(I(\varepsilon_1, \dots, \varepsilon_{n_1+(k_1-2)\lfloor\phi(n_1)\rfloor+\lfloor\phi(n_1)\rfloor-m})) \\ &\leq \frac{1}{\beta-1} \cdot \frac{1}{\beta^{\lfloor\phi(n_1)\rfloor-m-1}} \cdot \nu(I(\varepsilon_1, \dots, \varepsilon_{n_1+(k_1-2)\lfloor\phi(n_1)\rfloor+\lfloor\phi(n_1)\rfloor-m})) \end{aligned} \quad (4.10)$$

since $|I(\varepsilon_1, \dots, \varepsilon_{n_1+(k_1-1)\lfloor\phi(n_1)\rfloor+\lfloor\phi(n_1)\rfloor-m})| \leq \beta^{-(n_1+(k_1-1)\lfloor\phi(n_1)\rfloor+\lfloor\phi(n_1)\rfloor-m)}$. Repeating the above procedure, we have that

$$\nu(I(\varepsilon_1, \dots, \varepsilon_{n_1+(k_1-1)\lfloor\phi(n_1)\rfloor+\lfloor\phi(n_1)\rfloor-m})) \leq \left(\frac{1}{\beta-1} \cdot \frac{1}{\beta^{\lfloor\phi(n_1)\rfloor-m-1}} \right)^{k_1-1} \cdot \nu(I(\varepsilon_1, \dots, \varepsilon_{n_1})). \quad (4.11)$$

Combining this with (4.9) and (4.10), we obtain that

$$\begin{aligned} \nu(I(\varepsilon_1, \dots, \varepsilon_{n_2})) &\leq \frac{1}{\beta-1} \cdot \left(\frac{1}{\beta-1} \cdot \frac{1}{\beta^{\lfloor\phi(n_1)\rfloor-m-1}} \right)^{k_1-1} \cdot \nu(I(\varepsilon_1, \dots, \varepsilon_{n_1})) \\ &\leq \frac{\beta^{k_1(m+1-\log_\beta(\beta-1))+\lfloor\phi(n_1)\rfloor}}{\beta^{k_1\lfloor\phi(n_1)\rfloor}} \cdot \nu(I(\varepsilon_1, \dots, \varepsilon_{n_1})). \end{aligned}$$

Note that $m+1-\log_\beta(\beta-1) > 0$ and $k_1\lfloor\phi(n_1)\rfloor = n_2 - n_1 - r_1 - 1$, we have

$$\nu(I(\varepsilon_1, \dots, \varepsilon_{n_2})) \leq \frac{\beta^{(m+1-\log_\beta(\beta-1))(k_1+1)+\lfloor\phi(n_1)\rfloor+r_1+1}}{\beta^{n_2-n_1}} \cdot \nu(I(\varepsilon_1, \dots, \varepsilon_{n_1})).$$

• $r_1 > m$.

By the construction of \mathcal{D}_n , we know $\varepsilon_{n_2} \neq 0$ and $\varepsilon_{n_2-m} = \dots = \varepsilon_{n_2-1} = 0$. So,

$$\nu(I(\varepsilon_1, \dots, \varepsilon_{n_2})) = \frac{|I(\varepsilon_1, \dots, \varepsilon_{n_2})|}{\beta^{n_2-1} - \beta^{n_2}} \cdot \nu(I(\varepsilon_1, \dots, \varepsilon_{n_2-m-1})). \quad (4.12)$$

Notice that $\varepsilon_{n_2-m-1} \neq 0$, by the distribution of the measure ν , we deduce that

$$\begin{aligned} \nu(I(\varepsilon_1, \dots, \varepsilon_{n_2-m-1})) &= \frac{|I(\varepsilon_1, \dots, \varepsilon_{n_2-m-1})|}{\beta^{-(n_1+k_1\lfloor\phi(n_1)\rfloor)} - \beta^{-(n_1+k_1\lfloor\phi(n_1)\rfloor+1)}} \\ &\quad \times \nu(I(\varepsilon_1, \dots, \varepsilon_{n_1+(k_1-1)\lfloor\phi(n_1)\rfloor+\lfloor\phi(n_1)\rfloor-m})). \end{aligned} \quad (4.13)$$

It follows from (4.12) and (4.13) that

$$\nu(I(\varepsilon_1, \dots, \varepsilon_{n_2})) \leq \left(\frac{1}{\beta-1} \right)^2 \cdot \nu(I(\varepsilon_1, \dots, \varepsilon_{n_1+(k_1-1)\lfloor\phi(n_1)\rfloor+\lfloor\phi(n_1)\rfloor-m})).$$

Combining this with (4.11), we deduce that

$$\nu(I(\varepsilon_1, \dots, \varepsilon_{n_2})) \leq \left(\frac{1}{\beta-1} \right)^2 \cdot \left(\frac{1}{\beta-1} \cdot \frac{1}{\beta^{\lfloor\phi(n_1)\rfloor-m-1}} \right)^{k_1-1} \cdot \nu(I(\varepsilon_1, \dots, \varepsilon_{n_1})).$$

Since $m+1-\log_\beta(\beta-1) > 0$ and $k_1\lfloor\phi(n_1)\rfloor = n_2 - n_1 - r_1 - 1$, we obtain the desired result by following the calculations at the end of the case of $r_1 \leq m$.

As mentioned above, no matter $r_1 \leq m$ and $r_1 > m$, we always have

$$\nu(I(\varepsilon_1, \dots, \varepsilon_{n_2})) \leq \frac{\beta^{(m+1-\log_\beta(\beta-1))(k_1+1)+\lfloor\phi(n_1)\rfloor+r_1+1}}{\beta^{n_2-n_1}} \cdot \nu(I(\varepsilon_1, \dots, \varepsilon_{n_1})).$$

More generally, we have the following lemma whose proof is similar to the above arguments.

Lemma 4.4. For any $i \geq 1$,

$$\nu(I(\varepsilon_1, \dots, \varepsilon_{n_i})) \leq \beta^{U_i - n_i},$$

where $U_1 := 0$ and

$$U_i = (m+1 - \log_\beta(\beta-1)) \sum_{j=1}^{i-1} (k_j + 1) + \sum_{j=1}^{i-1} (\lfloor \phi(n_j) \rfloor + r_j + 1) \text{ for any } i \geq 2.$$

By Lemma 4.4, a simple calculation implies the following result.

Lemma 4.5. For any $i \geq 1$ and $1 \leq j \leq k_i - 1$,

$$\nu(I(\varepsilon_1, \dots, \varepsilon_{n_i+(j-1)\lfloor \phi(n_i) \rfloor + \lfloor \phi(n_i) \rfloor - m})) \leq \frac{\beta^{(m+1 - \log_\beta(\beta-1))(j-1) + U_i}}{\beta^{n_i+(j-1)\lfloor \phi(n_i) \rfloor}},$$

where U_i is as defined in Lemma 4.4.

Proof. Since $\varepsilon_{n_i+(j-1)\lfloor \phi(n_i) \rfloor + 1} \neq 0$, by the definition of the measure ν , we deduce

$$\begin{aligned} \nu(I(\varepsilon_1, \dots, \varepsilon_{n_i+(j-1)\lfloor \phi(n_i) \rfloor + \lfloor \phi(n_i) \rfloor - m})) &= \frac{|I(\varepsilon_1, \dots, \varepsilon_{n_i+(j-1)\lfloor \phi(n_i) \rfloor + \lfloor \phi(n_i) \rfloor - m})|}{\beta^{-(n_i+(j-1)\lfloor \phi(n_i) \rfloor)} - \beta^{-(n_i+(j-1)\lfloor \phi(n_i) \rfloor + 1)}} \\ &\quad \times \nu(I(\varepsilon_1, \dots, \varepsilon_{n_i+(j-2)\lfloor \phi(n_i) \rfloor + \lfloor \phi(n_i) \rfloor - m})). \end{aligned}$$

Therefore,

$$\nu(I(\varepsilon_1, \dots, \varepsilon_{n_i+(j-1)\lfloor \phi(n_i) \rfloor + \lfloor \phi(n_i) \rfloor - m})) \leq \frac{1}{\beta - 1} \cdot \frac{1}{\beta^{\lfloor \phi(n_i) \rfloor - m - 1}} \cdot \nu(I(\varepsilon_1, \dots, \varepsilon_{n_i+(j-2)\lfloor \phi(n_i) \rfloor + \lfloor \phi(n_i) \rfloor - m})).$$

Repeating the above procedure, we obtain that

$$\nu(I(\varepsilon_1, \dots, \varepsilon_{n_i+(j-1)\lfloor \phi(n_i) \rfloor + \lfloor \phi(n_i) \rfloor - m})) \leq \left(\frac{1}{\beta - 1} \cdot \frac{1}{\beta^{\lfloor \phi(n_i) \rfloor - m - 1}} \right)^{j-1} \cdot \nu(I(\varepsilon_1, \dots, \varepsilon_{n_i})).$$

By Lemma 4.4, we complete the proof. \square

Next we will check the inequality in Proposition 4.2 according to the classification of n at the end of Step 2. That is,

$$\nu(I(\varepsilon_1, \dots, \varepsilon_n)) \leq C_\delta \cdot |(\varepsilon_1, \dots, \varepsilon_n)|^{s-\delta}, \quad (4.14)$$

where $C_\delta > 0$ is a constant only depending on δ .

(I) When $n_i < n \leq n_i + \lfloor \phi(n_i) \rfloor$ ($i \geq 1$), by Lemma 4.4, we have that

$$\nu(I(\varepsilon_1, \dots, \varepsilon_n)) = \nu(I(\varepsilon_1, \dots, \varepsilon_{n_i})) \leq \beta^{U_i - n_i} \leq \beta^{\frac{\delta}{2}(n_i + \lfloor \phi(n_i) \rfloor) - n_i},$$

where the last inequality is from (4.4). It follows from Proposition 2.8 that

$$C_0 \beta^{-n} \leq |(\varepsilon_1, \dots, \varepsilon_n)| \leq \beta^{-n}$$

and hence

$$\frac{\nu(I(\varepsilon_1, \dots, \varepsilon_n))}{|(\varepsilon_1, \dots, \varepsilon_n)|^{s-\delta}} \leq C_0^{\delta-s} \beta^{n(s-\delta) + \frac{\delta}{2}(n_i + \lfloor \phi(n_i) \rfloor) - n_i} \leq C_0^{\delta-s} \beta^{n_i t_i},$$

where $t_i = (1 + \lfloor \phi(n_i) \rfloor / n_i)(s - \delta/2) - 1$ and the last inequality is because $n \leq n_i + \lfloor \phi(n_i) \rfloor$. By the definition of s and t_i , we know that $\lim_{i \rightarrow \infty} t_i = -\delta/(2s) \leq -\delta/2$ for (4.3). So there exists a constant C_1 such that $\beta^{n_i t_i} \leq C_1$ for all $i \geq 1$. Let $C_\delta = C_1 \cdot C_0^{\delta-s}$. Then we obtain that (4.14) holds for the measure ν and such a constant C_δ .

(II) When $n_i + j\lfloor \phi(n_i) \rfloor < n \leq n_i + j\lfloor \phi(n_i) \rfloor + \lfloor \phi(n_i) \rfloor - m$ ($i \geq 1$ and $1 \leq j \leq k_i - 1$), we deduce that

$$\nu(I(\varepsilon_1, \dots, \varepsilon_n)) = \frac{|I(\varepsilon_1, \dots, \varepsilon_n)|}{\beta^{-(n_i+j\lfloor \phi(n_i) \rfloor)} - \beta^{-(n_i+j\lfloor \phi(n_i) \rfloor + 1)}} \cdot \nu(I(\varepsilon_1, \dots, \varepsilon_{n_i+(j-1)\lfloor \phi(n_i) \rfloor + \lfloor \phi(n_i) \rfloor - m})).$$

In view of Lemma 4.5, we have that

$$\begin{aligned}\nu(I(\varepsilon_1, \dots, \varepsilon_n)) &\leq \frac{\beta^{n_i+j\lfloor\phi(n_i)\rfloor+1}}{\beta-1} \cdot \frac{1}{\beta^n} \cdot \frac{\beta^{(m+1-\log_\beta(\beta-1))(j-1)+U_i}}{\beta^{n_i+(j-1)\lfloor\phi(n_i)\rfloor}} \\ &= \frac{\beta^{-n}}{\beta-1} \cdot \beta^{(m+1-\log_\beta(\beta-1))(j-1)+U_i+\lfloor\phi(n_i)\rfloor+1}.\end{aligned}$$

Therefore,

$$\frac{\nu(I(\varepsilon_1, \dots, \varepsilon_n))}{|(\varepsilon_1, \dots, \varepsilon_n)|^{s-\delta}} \leq \frac{C_0^{\delta-s}}{\beta-1} \cdot \beta^{n(s-\delta-1)} \cdot \beta^{(m+1-\log_\beta(\beta-1))(j-1)+U_i+\lfloor\phi(n_i)\rfloor+1}. \quad (4.15)$$

Note that $s - \delta - 1 < 0$ and $n_i + j\lfloor\phi(n_i)\rfloor < n \leq n_i + j\lfloor\phi(n_i)\rfloor + \lfloor\phi(n_i)\rfloor - m$, so

$$\begin{aligned}&n(s - \delta - 1) + (m + 1 - \log_\beta(\beta - 1))(j - 1) + U_i + \lfloor\phi(n_i)\rfloor + 1 \\ &\leq (n_i + j\lfloor\phi(n_i)\rfloor)(s - \delta - 1) + (m + 1 - \log_\beta(\beta - 1))(j - 1) + U_i + \lfloor\phi(n_i)\rfloor + 1 \\ &= (n_i + \lfloor\phi(n_i)\rfloor)(s - \delta - 1) + (j - 1)W_i + U_i + \lfloor\phi(n_i)\rfloor + 1 \\ &\leq (n_i + \lfloor\phi(n_i)\rfloor)(s - \delta/2 - 1) + \lfloor\phi(n_i)\rfloor + 1,\end{aligned}$$

where $W_i = \lfloor\phi(n_i)\rfloor(s - \delta - 1) + m + 1 - \log_\beta(\beta - 1) < 0$ and the last inequality follows from the condition (4.4), i.e., $U_i \leq \delta(n_i + \lfloor\phi(n_i)\rfloor)/2$. Combining this with (4.15), we obtain that

$$\frac{\nu(I(\varepsilon_1, \dots, \varepsilon_n))}{|(\varepsilon_1, \dots, \varepsilon_n)|^{s-\delta}} \leq \frac{C_0^{\delta-s}}{\beta-1} \cdot \beta^{n_i t_i},$$

where $t_i = (1 + \lfloor\phi(n_i)\rfloor/n_i)(s - \delta/2 - 1) + (\lfloor\phi(n_i)\rfloor + 1)/n_i$. By (4.3) and the definition of s , we know that $\lim_{i \rightarrow \infty} t_i = -\delta/(2s) \leq -\delta/2$. The similar methods at the end of Case (I) assure that (4.14) holds.

(III) When $n_i + j\lfloor\phi(n_i)\rfloor + \lfloor\phi(n_i)\rfloor - m < n \leq n_i + (j + 1)\lfloor\phi(n_i)\rfloor$ ($i \geq 1$ and $1 \leq j \leq k_i - 1$), then

$$\nu(I(\varepsilon_1, \dots, \varepsilon_n)) = \nu(I(\varepsilon_1, \dots, \varepsilon_{n_i+j\lfloor\phi(n_i)\rfloor+\lfloor\phi(n_i)\rfloor-m})).$$

It follows from Lemma 4.5 that

$$\nu(I(\varepsilon_1, \dots, \varepsilon_{n_i+j\lfloor\phi(n_i)\rfloor+\lfloor\phi(n_i)\rfloor-m})) \leq \frac{\beta^{j(m+1-\log_\beta(\beta-1))+U_i}}{\beta^{n_i+j\lfloor\phi(n_i)\rfloor}}.$$

Therefore,

$$\frac{\nu(I(\varepsilon_1, \dots, \varepsilon_n))}{|(\varepsilon_1, \dots, \varepsilon_n)|^{s-\delta}} \leq C_0^{\delta-s} \beta^{n(s-\delta)} \cdot \frac{\beta^{j(m+1-\log_\beta(\beta-1))+U_i}}{\beta^{n_i+j\lfloor\phi(n_i)\rfloor}}. \quad (4.16)$$

Since $s - \delta > 0$ and $n_i + j\lfloor\phi(n_i)\rfloor + \lfloor\phi(n_i)\rfloor - m < n \leq n_i + (j + 1)\lfloor\phi(n_i)\rfloor$, we have

$$\begin{aligned}&n(s - \delta) + j(m + 1 - \log_\beta(\beta - 1)) + U_i - (n_i + j\lfloor\phi(n_i)\rfloor) \\ &\leq (n_i + (j + 1)\lfloor\phi(n_i)\rfloor)(s - \delta) + j(m + 1 - \log_\beta(\beta - 1)) + U_i - (n_i + j\lfloor\phi(n_i)\rfloor) \\ &= (n_i + \lfloor\phi(n_i)\rfloor)(s - \delta) + jW_i + U_i - n_i \leq (n_i + \lfloor\phi(n_i)\rfloor)(s - \delta/2) - n_i.\end{aligned}$$

Combining this with (4.16), we obtain that

$$\frac{\nu(I(\varepsilon_1, \dots, \varepsilon_n))}{|(\varepsilon_1, \dots, \varepsilon_n)|^{s-\delta}} \leq C_0^{\delta-s} \beta^{n_i t_i},$$

where $t_i = (1 + \lfloor\phi(n_i)\rfloor/n_i)(s - \delta/2) - 1$. By (4.3) and the definition of s , we know that $\lim_{i \rightarrow \infty} t_i = -\delta/(2s) \leq -\delta/2$. The similar methods at the end of Case (I) guarantee that (4.14) holds.

(IV) Let $n_i + k_i\lfloor\phi(n_i)\rfloor < n \leq n_{i+1}$ ($i \geq 1$).

(i) If $r_i \leq m$, we know that

$$\nu(I(\varepsilon_1, \dots, \varepsilon_n)) = \nu(I(\varepsilon_1, \dots, \varepsilon_{n_i+(k_i-1)\lfloor\phi(n_i)\rfloor+\lfloor\phi(n_i)\rfloor-m})).$$

It follows from Lemma 4.5 that

$$\nu(I(\varepsilon_1, \dots, \varepsilon_{n_i+(k_i-1)\lfloor\phi(n_i)\rfloor+\lfloor\phi(n_i)\rfloor-m})) \leq \frac{\beta^{(m+1-\log_\beta(\beta-1))(k_i-1)+U_i}}{\beta^{n_i+(k_i-1)\lfloor\phi(n_i)\rfloor}}.$$

Therefore,

$$\frac{\nu(I(\varepsilon_1, \dots, \varepsilon_n))}{|(\varepsilon_1, \dots, \varepsilon_n)|^{s-\delta}} \leq C_0^{\delta-s} \beta^{n(s-\delta)} \cdot \frac{\beta^{(m+1-\log_\beta(\beta-1))(k_i-1)+U_i}}{\beta^{n_i+(k_i-1)\lfloor\phi(n_i)\rfloor}}. \quad (4.17)$$

Notice that $n_i + k_i \lfloor\phi(n_i)\rfloor < n \leq n_{i+1}$ ($i \geq 1$) and $r_i \leq m$, by the definition of k_i , we know that $n_{i+1} = n_i + k_i \lfloor\phi(n_i)\rfloor + r_i + 1 \leq n_i + k_i \lfloor\phi(n_i)\rfloor + m + 1$. So,

$$\begin{aligned} & n(s-\delta) + (m+1-\log_\beta(\beta-1))(k_i-1) + U_i - (n_i + (k_i-1)\lfloor\phi(n_i)\rfloor) \\ & \leq (n_i + k_i \lfloor\phi(n_i)\rfloor + m+1)(s-\delta) + (m+1-\log_\beta(\beta-1))(k_i-1) + U_i - (n_i + (k_i-1)\lfloor\phi(n_i)\rfloor) \\ & = (n_i + \lfloor\phi(n_i)\rfloor)(s-\delta) + (m+1)(s-\delta) + (k_i-1)W_i + U_i - n_i \\ & \leq (n_i + \lfloor\phi(n_i)\rfloor)(s-\delta/2) - n_i + (m+1)(s-\delta), \end{aligned}$$

Combining this with (4.17), we have

$$\frac{\nu(I(\varepsilon_1, \dots, \varepsilon_n))}{|(\varepsilon_1, \dots, \varepsilon_n)|^{s-\delta}} \leq C_0^{\delta-s} \beta^{n_i t_i},$$

where $t_i = (1 + \lfloor\phi(n_i)\rfloor/n_i)(s-\delta/2) - 1 + (m+1)(s-\delta)/n_i$. By (4.3) and the definition of s , we know that $\lim_{i \rightarrow \infty} t_i = -\delta/(2s) \leq -\delta/2$ and hence that (4.14) holds.

(ii) Let $r_i > m$.

(1) For $n_i + k_i \lfloor\phi(n_i)\rfloor < n \leq n_{i+1} - m - 1$, we obtain that

$$\nu(I(\varepsilon_1, \dots, \varepsilon_n)) = \frac{|I(\varepsilon_1, \dots, \varepsilon_n)|}{\beta^{-(n_i+k_i\lfloor\phi(n_i)\rfloor)} - \beta^{-(n_i+k_i\lfloor\phi(n_i)\rfloor+1)}} \cdot \nu(I(\varepsilon_1, \dots, \varepsilon_{n_i+(k_i-1)\lfloor\phi(n_i)\rfloor+\lfloor\phi(n_i)\rfloor-m})).$$

It follows from Lemma 4.5 that

$$\nu(I(\varepsilon_1, \dots, \varepsilon_{n_i+(k_i-1)\lfloor\phi(n_i)\rfloor+\lfloor\phi(n_i)\rfloor-m})) \leq \frac{\beta^{(m+1-\log_\beta(\beta-1))(k_i-1)+U_i}}{\beta^{n_i+(k_i-1)\lfloor\phi(n_i)\rfloor}}.$$

So,

$$\begin{aligned} \nu(I(\varepsilon_1, \dots, \varepsilon_n)) & \leq \frac{\beta^{n_i+k_i\lfloor\phi(n_i)\rfloor+1}}{\beta-1} \cdot \frac{1}{\beta^n} \cdot \frac{\beta^{(m+1-\log_\beta(\beta-1))(k_i-1)+U_i}}{\beta^{n_i+(k_i-1)\lfloor\phi(n_i)\rfloor}} \\ & = \frac{\beta^{-n}}{\beta-1} \cdot \beta^{(m+1-\log_\beta(\beta-1))(k_i-1)+U_i+\lfloor\phi(n_i)\rfloor+1}. \end{aligned}$$

Therefore,

$$\frac{\nu(I(\varepsilon_1, \dots, \varepsilon_n))}{|(\varepsilon_1, \dots, \varepsilon_n)|^{s-\delta}} \leq \frac{C_0^{\delta-s}}{\beta-1} \cdot \beta^{n(s-\delta-1)} \cdot \beta^{(m+1-\log_\beta(\beta-1))(k_i-1)+U_i+\lfloor\phi(n_i)\rfloor+1}. \quad (4.18)$$

Note that $s-\delta-1 < 0$ and $n_i + k_i \lfloor\phi(n_i)\rfloor < n \leq n_{i+1} - m - 1$, we deduce

$$\begin{aligned} & n(s-\delta-1) + (m+1-\log_\beta(\beta-1))(k_i-1) + U_i + \lfloor\phi(n_i)\rfloor + 1 \\ & \leq (n_i + k_i \lfloor\phi(n_i)\rfloor)(s-\delta-1) + (m+1-\log_\beta(\beta-1))(k_i-1) + U_i + \lfloor\phi(n_i)\rfloor + 1 \\ & \leq (n_i + \lfloor\phi(n_i)\rfloor)(s-\delta-1) + (k_i-1)W_i + U_i + \lfloor\phi(n_i)\rfloor + 1 \\ & \leq (n_i + \lfloor\phi(n_i)\rfloor)(s-\delta/2-1) + \lfloor\phi(n_i)\rfloor + 1. \end{aligned}$$

Combining this with (4.18), we obtain that

$$\frac{\nu(I(\varepsilon_1, \dots, \varepsilon_n))}{|(\varepsilon_1, \dots, \varepsilon_n)|^{s-\delta}} \leq \frac{C_0^{\delta-s}}{\beta-1} \cdot \beta^{n_i t_i},$$

where $t_i = (1 + \lfloor\phi(n_i)\rfloor/n_i)(s-\delta/2-1) + (\lfloor\phi(n_i)\rfloor+1)/n_i$. By (4.3) and the definition of s , we know that $\lim_{i \rightarrow \infty} t_i = -\delta/(2s) \leq -\delta/2$ and hence that (4.14) holds.

(2) For $n_{i+1} - m - 1 < n \leq n_{i+1}$, we get that

$$\nu(I(\varepsilon_1, \dots, \varepsilon_n)) = \nu(I(\varepsilon_1, \dots, \varepsilon_{n_{i+1}-m-1})).$$

Since $\varepsilon_{n_i+k_i\lfloor\phi(n_i)\rfloor+1} \neq 0$, by the distribution of the measure ν , we deduce

$$\nu(I(\varepsilon_1, \dots, \varepsilon_{n_{i+1}-m-1})) = \frac{|I(\varepsilon_1, \dots, \varepsilon_{n_{i+1}-m-1})|}{\beta^{-(n_i+k_i\lfloor\phi(n_i)\rfloor)} - \beta^{-(n_i+k_i\lfloor\phi(n_i)\rfloor+1)}} \cdot \nu(I(\varepsilon_1, \dots, \varepsilon_{n_i+(k_i-1)\lfloor\phi(n_i)\rfloor+\lfloor\phi(n_i)\rfloor-m})).$$

Therefore,

$$\frac{\nu(I(\varepsilon_1, \dots, \varepsilon_{n_{i+1}-m-1}))}{\nu(I(\varepsilon_1, \dots, \varepsilon_{n_i+(k_i-1)\lfloor\phi(n_i)\rfloor+\lfloor\phi(n_i)\rfloor-m}))} \leq \frac{\beta^{n_i+k_i\lfloor\phi(n_i)\rfloor+1}}{\beta-1} \cdot \frac{1}{\beta^{n_{i+1}-m-1}} = \frac{1}{\beta-1} \cdot \frac{1}{\beta^{r_i-m-1}}, \quad (4.19)$$

where the last equation is from the definition of n_{i+1} , i.e., $n_{i+1} = n_i + k_i\lfloor\phi(n_i)\rfloor + r_i + 1$. Notice that

$$\nu(I(\varepsilon_1, \dots, \varepsilon_{n_i+(k_i-1)\lfloor\phi(n_i)\rfloor+\lfloor\phi(n_i)\rfloor-m})) \leq \frac{\beta^{(m+1-\log_\beta(\beta-1))(k_i-1)+U_i}}{\beta^{n_i+(k_i-1)\lfloor\phi(n_i)\rfloor}},$$

combining this with (4.19), we obtain that

$$\nu(I(\varepsilon_1, \dots, \varepsilon_{n_{i+1}-m-1})) \leq \frac{1}{\beta-1} \cdot \frac{1}{\beta^{r_i-m-1}} \cdot \frac{\beta^{(m+1-\log_\beta(\beta-1))(k_i-1)+U_i}}{\beta^{n_i+(k_i-1)\lfloor\phi(n_i)\rfloor}}.$$

Therefore,

$$\begin{aligned} \frac{\nu(I(\varepsilon_1, \dots, \varepsilon_n))}{|(\varepsilon_1, \dots, \varepsilon_n)|^{s-\varepsilon}} &\leq \frac{C_0^{\delta-s}}{\beta-1} \cdot \frac{\beta^{n(s-\delta)}}{\beta^{r_i-m-1}} \cdot \frac{\beta^{(m+1-\log_\beta(\beta-1))(k_i-1)+U_i}}{\beta^{n_i+(k_i-1)\lfloor\phi(n_i)\rfloor}} \\ &\leq \frac{C_0^{\delta-s}}{\beta-1} \cdot \frac{\beta^{(n_i+k_i\lfloor\phi(n_i)\rfloor+r_i+1)(s-\delta)+(m+1-\log_\beta(\beta-1))(k_i-1)+U_i}}{\beta^{n_i+(k_i-1)\lfloor\phi(n_i)\rfloor+l_i-m-1}} \\ &= \frac{C_0^{\delta-s}}{\beta-1} \cdot \beta^{(n_i+\lfloor\phi(n_i)\rfloor)(s-\delta)+(r_i+1)(s-\delta-1)+(k_i-1)W_i+U_i-n_i+(m+2)} \\ &\leq \frac{C_0^{\delta-s}}{\beta-1} \cdot \beta^{(n_i+\lfloor\phi(n_i)\rfloor)(s-\delta/2)-n_i+(m+2)} = \frac{C_0^{\delta-s}}{\beta-1} \cdot \beta^{n_i t_i}, \end{aligned}$$

where $t_i = (1 + \lfloor\phi(n_i)\rfloor/n_i)(s-\delta/2) - 1 + (m+2)/n_i$, the second inequality follows from $s-\delta > 0$ and $n \leq n_{i+1} = n_i + k_i\lfloor\phi(n_i)\rfloor + r_i + 1$ and the last inequality is from the condition (4.4), $s-\delta-1 < 0$ and $W_i < 0$. By (4.3) and the definition of s , we know that $\lim_{i \rightarrow \infty} t_i = -\delta/(2s) \leq -\delta/2$ and hence that (4.14) holds.

Applying Proposition 4.2 to the set F and the measure ν , we have that

$$\dim_H F_\phi \geq \dim_H F \geq s - \delta.$$

Letting $\delta \rightarrow 0^+$, we have $\dim_H F_\phi \geq s$. We eventually obtain

$$\dim_H F_\phi \geq \frac{1}{1 + \lim_{n \rightarrow \infty} \phi(n)/n}.$$

4.2.2. General case for any $\beta > 1$. Now we will extend the result of Lemma 4.3 from $\beta \in A_0$ to any $\beta > 1$ by using the approximation method for the β -shift in Section 2.2. Let $\beta > \beta' > 1$ and $\beta' \in A_0$. Here we denote $\ell_n(x, \beta)$ and $\ell_n(x, \beta')$ the length of the longest string of zeros just after the n -th digit in the β -expansion and β' -expansion of x respectively. Lemma 4.3 has showed that

$$\dim_H \left\{ x \in [0, 1) : \limsup_{n \rightarrow \infty} \frac{\ell_n(x, \beta')}{\phi(n)} = 1 \right\} \geq \frac{1}{1 + \liminf_{n \rightarrow \infty} \phi(n)/n} \quad (4.20)$$

under the assumption that ϕ is a positive and nondecreasing function with $\phi(n) \rightarrow \infty$ as $n \rightarrow \infty$.

Lemma 4.6. *Let $\beta > 1$ be a real number. Assume that ϕ is a positive and nondecreasing function defined on \mathbb{N} with $\phi(n) \rightarrow \infty$ as $n \rightarrow \infty$. Then*

$$\dim_H F_\phi \geq \frac{1}{1 + \liminf_{n \rightarrow \infty} \phi(n)/n}.$$

Proof. Let $\beta > \beta' > 1$ and $\beta' \in A_0$. Since $H_\beta^{\beta'} := \pi_\beta(\Sigma_{\beta'})$ is a Cantor set of $[0, 1)$, we have that

$$\dim_H \left\{ x \in [0, 1) : \limsup_{n \rightarrow \infty} \frac{\ell_n(x, \beta)}{\phi(n)} = 1 \right\} \geq \dim_H \left\{ x \in H_\beta^{\beta'} : \limsup_{n \rightarrow \infty} \frac{\ell_n(x, \beta)}{\phi(n)} = 1 \right\}. \quad (4.21)$$

It follows from Proposition 2.9 (iv) that the function h is Hölder continuous and hence that

$$\dim_{\text{H}} \left\{ x \in H_{\beta}^{\beta'} : \limsup_{n \rightarrow \infty} \frac{\ell_n(x, \beta)}{\phi(n)} = 1 \right\} \geq \frac{\log \beta'}{\log \beta} \cdot \dim_{\text{H}} h \left(\left\{ x \in H_{\beta}^{\beta'} : \limsup_{n \rightarrow \infty} \frac{\ell_n(x, \beta)}{\phi(n)} = 1 \right\} \right). \quad (4.22)$$

Note that $\varepsilon(h(x), \beta') = \varepsilon(x, \beta)$ for any $x \in H_{\beta}^{\beta'}$, we deduce that $\ell_n(h(x), \beta') = \ell_n(x, \beta)$ by the definition of $\ell_n(x, \beta)$ defined in Section 3. Since h is bijective, we obtain

$$h \left(\left\{ x \in H_{\beta}^{\beta'} : \limsup_{n \rightarrow \infty} \frac{\ell_n(x, \beta)}{\phi(n)} = 1 \right\} \right) = \left\{ y \in [0, 1) : \limsup_{n \rightarrow \infty} \frac{\ell_n(y, \beta')}{\phi(n)} = 1 \right\}.$$

Combining this with (4.21) and (4.22), we have that

$$\dim_{\text{H}} F_{\phi} \geq \frac{\log \beta'}{\log \beta} \cdot \dim_{\text{H}} \left\{ y \in [0, 1) : \limsup_{n \rightarrow \infty} \frac{\ell_n(y, \beta')}{\phi(n)} = 1 \right\} \geq \frac{\log \beta'}{\log \beta} \cdot \frac{1}{1 + \liminf_{n \rightarrow \infty} \phi(n)/n},$$

where the last inequality is from (4.20). At last, let $\beta' \rightarrow \beta$, we complete the proof since A_0 is dense in $(1, +\infty)$. \square

By the upper bound Hausdorff dimension of F_{ϕ} in Section 4.1, we have

Proposition 4.7. *Let $\beta > 1$ be a real number. Assume that ϕ is a positive and nondecreasing function defined on \mathbb{N} with $\phi(n) \rightarrow \infty$ as $n \rightarrow \infty$. Then*

$$\dim_{\text{H}} F_{\phi} = \frac{1}{1 + \liminf_{n \rightarrow \infty} \phi(n)/n}.$$

For any $\beta \in A_0$, being similar to the Lemma 4.3, we can construct a Cantor-like subset E of E_{ϕ} . We choose a subsequence $\{n_i\}_{i \geq 1}$ such that $\liminf_{n \rightarrow \infty} \frac{\phi(n)}{n} = \lim_{i \rightarrow \infty} \frac{\phi(n_i)}{n_i}$ and $n_{i+1} > n_i + \lfloor \phi(n_i) \rfloor + 1$. Denote by \mathcal{C}_n the set of n -th order cylinders $I(\varepsilon_1, \dots, \varepsilon_n)$ satisfying $\varepsilon_k = 0$ if $k = n_i + j$ for $i \geq 1$ and $1 \leq j \leq \lfloor \phi(n_i) \rfloor + 1$; otherwise, $\varepsilon_k \in \mathcal{A}$. Let

$$E = \bigcap_{n=1}^{\infty} \bigcup_{I(\varepsilon_1, \dots, \varepsilon_n) \in \mathcal{C}_n} I(\varepsilon_1, \dots, \varepsilon_n).$$

Then $E \subset E_{\phi}$. We can also define a measure supported on E like the measure ν on F . Then it can be shown that this measure satisfies the modified mass distributed principle (i.e., Proposition 4.2) by similar skills in the proof of Lemma 4.3. Thus, we get a lower bound of the Hausdorff dimension of E_{ϕ} . Finally, we use the approximation method for the β -shift in Section 2.2 to extend this result from $\beta \in A_0$ to any $\beta > 1$. Combing this with Lemma 4.1, we have the following.

Proposition 4.8. *Let $\beta > 1$ be a real number. Assume that ϕ is a nonnegative function defined on \mathbb{N} . Then*

$$\dim_{\text{H}} E_{\phi} = \frac{1}{1 + \liminf_{n \rightarrow \infty} \phi(n)/n}.$$

Similar with Proposition 4.8, replacing $\phi(n)$ by $(\phi(n) - n)$, the following is obtained.

Proposition 4.9. *Let $\beta > 1$ be a real number. Assume that ϕ is a nonnegative function defined on \mathbb{N} satisfying $\liminf_{n \rightarrow \infty} \phi(n)/n \geq 1$. Then*

$$\dim_{\text{H}} \{x \in [0, 1) : \ell_n(x) \geq \phi(n) - n \text{ i.o.}\} = \frac{1}{\liminf_{n \rightarrow \infty} \phi(n)/n}.$$

Proof. Let $\liminf_{n \rightarrow \infty} \phi(n)/n \geq 1$. When $\liminf_{n \rightarrow \infty} (\phi(n) - n) > 0$, then $\phi(n) > n$ holds for sufficiently large n and hence the result is got via replacing $\phi(n)$ by $(\phi(n) - n)$ from Proposition 4.8. When $\liminf_{n \rightarrow \infty} (\phi(n) - n) < 0$, we know that $\phi(n) \leq n$ holds for infinitely many $n \in \mathbb{N}$ and hence $\liminf_{n \rightarrow \infty} \phi(n)/n = 1$. Then $\{x \in [0, 1) : \ell_n(x) \geq \phi(n) - n \text{ i.o.}\} = [0, 1)$ by the definition of $\ell_n(x)$ and hence the desired result is true. Now let $\liminf_{n \rightarrow \infty} (\phi(n) - n) = 0$. In this case, if $\limsup_{n \rightarrow \infty} (\phi(n) - n) > 0$, then $\phi(n) > n$ holds for infinitely many $n \in \mathbb{N}$ and hence the method of the construction of the Cantor-like set in Proposition

4.8 replacing $\phi(n)$ by $(\phi(n) - n)$ is valid; if $\limsup_{n \rightarrow \infty} (\phi(n) - n) \leq 0$, then $\lim_{n \rightarrow \infty} (\phi(n) - n) = 0$ and hence $\liminf_{n \rightarrow \infty} \phi(n)/n = 1$. By the definition of $\ell_n(x)$, we know $\{x \in [0, 1) : \ell_n(x) \geq \phi(n) - n \text{ i.o.}\} = [0, 1)$ and that the the desired result is obtained. \square

During the construction of the Cantor-like set F in Lemma 4.3, we can obtain the following lemma by replacing $\phi(n)$ by $(\phi(n) - n)$ and using the approximation method for the β -shift in Section 2.2.

Lemma 4.10. *Let $\beta > 1$ be a real number. Assume that ϕ is a positive and nondecreasing function defined on \mathbb{N} satisfying $\liminf_{n \rightarrow \infty} \phi(n)/n \geq 1$. Then*

$$\dim_H \left\{ x \in [0, 1) : \limsup_{n \rightarrow \infty} \frac{n + \ell_n(x)}{\phi(n)} = 1 \right\} \geq \frac{1}{\liminf_{n \rightarrow \infty} \phi(n)/n}.$$

Proof. When $\liminf_{n \rightarrow \infty} \phi(n)/n > 1$, we obtain that $\phi(n) > n$ holds for sufficiently large n and that $\liminf_{n \rightarrow \infty} (\phi(n) - n) = +\infty$. Note that ϕ is nondecreasing, so both the method of the construction of the Cantor-like set in the proof of Lemma 4.3 by replacing $\phi(n)$ by $(\phi(n) - n)$ and the approximation method for the β -shift in Section 2.2 are valid. Being similar to the proofs of Lemmas 4.3 and 4.6, we obtain the desired result. When $\liminf_{n \rightarrow \infty} \phi(n)/n = 1$, i.e., $\limsup_{n \rightarrow \infty} n/\phi(n) = 1$. Since $\frac{n + \ell_n(x)}{\phi(n)} = \frac{n}{\phi(n)} \cdot (1 + \frac{\ell_n(x)}{n})$, it is easy to see that

$$\left\{ x \in [0, 1) : \limsup_{n \rightarrow \infty} \frac{n + \ell_n(x)}{\phi(n)} = 1 \right\} \supseteq \left\{ x \in [0, 1) : \lim_{n \rightarrow \infty} \frac{\ell_n(x)}{n} = 0 \right\}.$$

Note that

$$\left\{ x \in [0, 1) : \limsup_{n \rightarrow \infty} \frac{\ell_n(x)}{n} = 0 \right\} \supseteq \left\{ x \in [0, 1) : \limsup_{n \rightarrow \infty} \frac{\ell_n(x)}{\log n} = 1 \right\},$$

by Lemma 4.6, we obtain that

$$\dim_H \left\{ x \in [0, 1) : \lim_{n \rightarrow \infty} \frac{n + \ell_n(x)}{\phi(n)} = 1 \right\} \geq 1.$$

\square

In view of Lemmas 4.1 and 4.10, we have

Proposition 4.11. *Let $\beta > 1$ be a real number. Assume that ϕ is a positive and nondecreasing function defined on \mathbb{N} satisfying $\liminf_{n \rightarrow \infty} \phi(n)/n \geq 1$. Then*

$$\dim_H \left\{ x \in [0, 1) : \limsup_{n \rightarrow \infty} \frac{n + \ell_n(x)}{\phi(n)} = 1 \right\} = \frac{1}{\liminf_{n \rightarrow \infty} \phi(n)/n}.$$

5. THE PROOF OF THEOREM 1.3

In this section, we will prove Theorem 1.3.

Proof of Theorem 1.3. (i) If $\eta = 0$, we have $\limsup_{n \rightarrow \infty} n/\phi(n) = +\infty$. By the definition of $\ell_n(x)$ defined in Section 3, we deduce that

$$\limsup_{n \rightarrow \infty} \frac{n + \ell_n(x)}{\phi(n)} \geq \limsup_{n \rightarrow \infty} \frac{n}{\phi(n)} = +\infty$$

for any $x \in [0, 1)$. It follows from the second inequality of (3.1) that

$$\limsup_{n \rightarrow \infty} \frac{n + \ell_n(x)}{\phi(n)} \leq -\liminf_{n \rightarrow \infty} \frac{1}{\phi(n)} \log_\beta(x - \omega_n(x)) \quad (5.1)$$

for any $x \in [0, 1)$. Thus,

$$\liminf_{n \rightarrow \infty} \frac{1}{\phi(n)} \log_\beta(x - \omega_n(x)) = -\infty$$

for any $x \in [0, 1)$. If $0 < \eta < 1$, by the definition of $\ell_n(x)$, we obtain that

$$\limsup_{n \rightarrow \infty} \frac{n + \ell_n(x)}{\phi(n)} \geq \limsup_{n \rightarrow \infty} \frac{n}{\phi(n)} = \frac{1}{\liminf_{n \rightarrow \infty} \phi(n)/n} = \frac{1}{\eta} > 1$$

for any $x \in [0, 1)$. In view of (5.1), we deduce that

$$\liminf_{n \rightarrow \infty} \frac{1}{\phi(n)} \log_\beta(x - \omega_n(x)) < -1 \quad \text{for any } x \in [0, 1).$$

Therefore, the set

$$B_\phi = \left\{ x \in [0, 1) : \liminf_{n \rightarrow \infty} \frac{1}{\phi(n)} \log_\beta(x - \omega_n(x)) = -1 \right\}$$

is empty.

(ii) Let $\eta \geq 1$. So $\phi(n) \rightarrow \infty$ as $n \rightarrow \infty$ and then it follows from (3.1) that

$$\limsup_{n \rightarrow \infty} \frac{n + \ell_n(x)}{\phi(n)} = - \liminf_{n \rightarrow \infty} \frac{1}{\phi(n)} \log_\beta(x - \omega_n(x))$$

for any $x \in [0, 1)$. Note that ϕ is nondecreasing, by Proposition 4.11, we obtain $\dim_H A_\phi = 1/\eta$. \square

6. APPLICATIONS

6.1. The orbits of real numbers under β -transformation. For any $x \in [0, 1)$, we say that the sequence $x, T_\beta x, \dots, T_\beta^n x, \dots$ is the orbit of x under T_β . In 2011, Li and Chen [24] studied the topological properties of the orbits of $x \in [0, 1)$ under T_β by considering the set

$$O_\beta = \{x \in [0, 1) : \text{the orbit of } x \text{ under } T_\beta \text{ is dense in } [0, 1]\}.$$

They proved that O_β is uncountable, dense, of the second category and has full Lebesgue measure, while its complementary set O_β^c is uncountable, dense, of the first category and has full Hausdorff dimension. They also showed that the β -transformation T_β is chaotic in the sense of both Li-Yorke and Devaney. Recently, Ban and Li [4] studied the multifractal spectra for the recurrence rate of the first return time of β -transformation, including the cases returning to the ball and cylinder.

For any $n \in \mathbb{N}$, by the definitions of $\omega_n(x)$ and T_β in Section 1, we obtain

$$x - \omega_n(x) = \frac{T_\beta^n x}{\beta^n}. \quad (6.1)$$

Combing this with (3.1), we have

$$\frac{1}{\beta^{\ell_n(x)+1}} \leq T_\beta^n x \leq \frac{1}{\beta^{\ell_n(x)}}. \quad (6.2)$$

As a consequence of Proposition 3.2, we give a quantitative formula for the growth speed of the orbit of x under T_β .

Theorem 6.1. *Let $\beta > 1$ be a real number. Then for λ -almost all $x \in [0, 1)$,*

$$\liminf_{n \rightarrow \infty} \frac{\log T_\beta^n x}{\log n} = -1.$$

Moreover, for any real number $x \in [0, 1)$ whose β -expansion is infinite, we have

$$\limsup_{n \rightarrow \infty} \frac{\log T_\beta^n x}{\log n} = 0.$$

By Proposition 4.7 and the inequalities (6.2), we obtain the Hausdorff dimension of exceptional set according to the above metric result.

Theorem 6.2. *Let $\beta > 1$ be a real number and ϕ be a positive and nondecreasing function defined on \mathbb{N} with $\phi(n) \rightarrow \infty$ as $n \rightarrow \infty$. Denote $\eta := \liminf_{n \rightarrow \infty} \phi(n)/n$. Then*

$$\dim_H \left\{ x \in [0, 1) : \liminf_{n \rightarrow \infty} \frac{\log T_\beta^n x}{\phi(n)} = -1 \right\} = \frac{1}{1 + \eta/\log \beta}.$$

As an direct application of Theorem 6.2, we obtain the Hausdorff dimensions of the level sets.

Corollary 6.3. *Let $\beta > 1$ be a real number. Then for any $\alpha \geq 0$,*

$$\dim_H \left\{ x \in [0, 1) : \liminf_{n \rightarrow \infty} \frac{\log T_\beta^n x}{\log n} = -\alpha \right\} = 1.$$

Proof. For $\alpha > 0$, taking $\phi(n) = \alpha \log n$, then $\phi(n)$ is positive and nondecreasing with $\phi(n) \rightarrow \infty$ as $n \rightarrow \infty$. Applying these $\phi(n)$ to Theorem 6.2, note that $\liminf_{n \rightarrow \infty} \phi(n)/n = 0$, we obtain the desired result. Now let $\alpha = 0$. Notice that

$$\left\{ x \in [0, 1) : \liminf_{n \rightarrow \infty} \frac{\log T_\beta^n x}{\log n} = 0 \right\} \supseteq \left\{ x \in [0, 1) : \liminf_{n \rightarrow \infty} \frac{\log T_\beta^n x}{\log \log n} = -1 \right\}$$

and the later set has full Hausdorff dimension by Theorem 6.2, so we complete the proof. \square

Application of Corollary 6.3 indicates that the set of points such that the metric result in Theorem 6.1 does not hold has full Hausdorff dimension.

6.2. The shrinking target type problem for β -transformations. The typical shrinking target problem with the given target aims at investigating the Hausdorff dimensions of sets of points whose orbits are close to some previously chosen point (see Hill and Velani [19]). In fact, Bugeaud and Wang [7] has given a more general result about this problem for β -expansions. Tan and Wang [38] concerned the quantitative recurrence properties of the beta dynamical system $([0, 1], T_\beta)$ for general $\beta > 1$ and the Hausdorff dimension of the set of points with the prescribed recurrence rate is determined. Liao and Seuret [28] studied the shrinking target problem for expanding Markov maps with a finite partition. Recently, Seuret and Wang [37] investigated a quantitative version of Poincaré's recurrence theorem in a conformal iterated function system, which includes the dynamical systems of p -adic expansions, continued fraction expansion, as well as some dynamical systems defined on fractal sets. For more applications about the shrinking target problem, see [22, 25, 26, 29].

By Proposition 4.8 and the inequalities (6.2), we have the following Shrinking target problem for β -transformations which is a special case of Bugeaud and Wang [7, Theorem 1.6].

Theorem 6.4 (Bugeaud and Wang [7]). *Let $\beta > 1$ be a real number and ϕ be a positive function defined on \mathbb{N} . Then*

$$\dim_H \left\{ x \in [0, 1) : T_\beta^n x \leq \beta^{-\phi(n)} \text{ i.o.} \right\} = \frac{1}{1 + \liminf_{n \rightarrow \infty} \phi(n)/n}.$$

Proof. It follows from (6.2) that

$$\{x \in [0, 1) : \ell_n(x) \geq \phi(n) \text{ i.o.}\} \subseteq \left\{ x \in [0, 1) : T_\beta^n x \leq \beta^{-\phi(n)} \text{ i.o.} \right\} \subseteq \{x \in [0, 1) : \ell_n(x) \geq \phi(n) - 1 \text{ i.o.}\}.$$

In view of Proposition 4.8, we know both left-hand and right-hand sets are of Hausdorff dimension $1/(1 + \liminf_{n \rightarrow \infty} \phi(n)/n)$ and hence we complete the proof. \square

6.3. The Diophantine-type problem for β -expansions. Notice that the classical Diophantine questions concern the dimension of the set

$$\{x \in [0, 1) : |x - p/q| \leq q^{-2\tau} \text{ for infinitely many couples } (p, q) = 1\},$$

the following is interpreted as a Diophantine approximation problem of β -expansions. In fact, in view of (6.1), this problem can also be viewed as a shrinking target problem for β -transformations.

Theorem 6.5 (Bugeaud and Wang [7]). *Let $\beta > 1$ be a real number and ϕ be a positive function defined on \mathbb{N} . Denote $\eta = \liminf_{n \rightarrow \infty} \phi(n)/n$ and*

$$C_\phi = \left\{ x \in [0, 1) : x - \omega_n(x) \leq \beta^{-\phi(n)} \text{ i.o.} \right\}$$

Then

- (i) *If $0 \leq \eta < 1$, then $\dim_H B_\phi = 1$.*
- (ii) *If $\eta \geq 1$, then $\dim_H B_\phi = 1/\eta$.*

Proof. In view of (3.1), we deduce that

$$\{x \in [0, 1) : \ell_n(x) \geq \phi(n) - n \text{ i.o.}\} \subseteq C_\phi \subseteq \{x \in [0, 1) : \ell_n(x) \geq \phi(n) - n - 1 \text{ i.o.}\}.$$

When $0 \leq \eta < 1$, we obtain $\phi(n) - n \leq 0$ for infinitely many $n \in \mathbb{N}$. Thus, by the definition of $\ell_n(x)$, the set of points x such that $\ell_n(x) \geq \phi(n) - n$ for infinitely many n is the unit interval $[0, 1)$ and hence that $\dim_{\mathbb{H}} B_\phi = 1$. When $\eta \geq 1$, it follows from Proposition 4.9 that $\dim_{\mathbb{H}} B_\phi = 1/\eta$. \square

6.4. The run-length function. Recall that

$$r_n(x) = \sup \{k \geq 0 : \varepsilon_{i+1}(x) = \cdots = \varepsilon_{i+k}(x) = 0 \text{ for some } 0 \leq i \leq n - k\},$$

i.e., the maximal length of consecutive zeros in the first n digits of the β -expansion of x . Let

$$G_\phi = \left\{ x \in [0, 1) : \limsup_{n \rightarrow \infty} \frac{r_n(x)}{\phi(n)} = 1 \right\},$$

where ϕ is a positive function defined on \mathbb{N} . Now let \mathcal{H} denote the set of positive and nondecreasing functions defined on \mathbb{N} satisfying $\phi(n) \rightarrow \infty$ as $n \rightarrow \infty$ and

$$\lim_{n \rightarrow \infty} \frac{\phi(n + \phi(n))}{\phi(n)} = 1. \quad (6.3)$$

It is noting that the following examples of ϕ are the typical elements of \mathcal{H} .

- $\phi(n) = \alpha n^\gamma$ with $\alpha > 0$ and $0 < \gamma < 1$.
- $\phi(n) = \alpha(\log n)^\gamma$ with $\alpha > 0$ and $\gamma > 0$.
- $\phi(n) = \alpha n/(\log n)^\gamma$ with $\alpha > 0$ and $\gamma > 0$.

The next two lemmas are the basic properties of the function in \mathcal{H} .

Lemma 6.6. *Let $\phi \in \mathcal{H}$. Then*

$$\lim_{n \rightarrow \infty} \frac{\phi(n + \delta\phi(n))}{\phi(n)} = 1$$

for any integer $\delta > 0$.

Proof. Let $\phi \in \mathcal{H}$. Then limit (6.3) holds. For any $\varepsilon > 0$, there exists $N := N_\varepsilon > 0$ such that for all $n \geq N_\varepsilon$, we have

$$\phi(n + \phi(n)) \leq (1 + \varepsilon)\phi(n). \quad (6.4)$$

For any integer $j > 0$, by the monotonic property of ϕ , we have $\phi(n + \phi(n)) \geq \phi(n)$ and hence that

$$\phi(n + j\phi(n) + \phi(n + j\phi(n))) \geq \phi(n + (j + 1)\phi(n)).$$

On the other hand, in view of (6.4), we obtain that

$$\phi(n + j\phi(n) + \phi(n + j\phi(n))) \leq (1 + \varepsilon)\phi(n + j\phi(n))$$

by regarding $(n + j\phi(n))$ as a whole. Therefore, for any integer j and $n \geq N$, we have

$$\phi(n + (j + 1)\phi(n)) \leq (1 + \varepsilon)\phi(n + j\phi(n)).$$

For any integer $\delta > 0$ and $n \geq N$, we obtain that

$$1 \leq \frac{\phi(n + \delta\phi(n))}{\phi(n)} = \prod_{j=0}^{\delta-1} \frac{\phi(n + (j + 1)\phi(n))}{\phi(n + j\phi(n))} \leq (1 + \varepsilon)^\delta$$

holds for any $\varepsilon > 0$, which implies the desired result. \square

Lemma 6.7. *Let $\phi \in \mathcal{H}$. Then $\liminf_{n \rightarrow \infty} \phi(n)/n = 0$.*

Proof. We assume that $\liminf_{n \rightarrow \infty} \phi(n)/n = \eta > 0$. Let $\{n_k\}_{k \geq 1}$ be a subsequence such that $\eta = \lim_{k \rightarrow \infty} \phi(n_k)/n_k$. For any $\varepsilon > 0$, there exists $K = K_\varepsilon > 0$ such that for all $k \geq K$, we have $\phi(n_k) \geq (\eta - \varepsilon)n_k$. By the monotonic property of ϕ , we deduce that

$$\phi(n_k + \phi(n_k)) \geq \phi(n_k + (\eta - \varepsilon)n_k) \geq (\eta - \varepsilon)(1 + \eta - \varepsilon)n_k$$

for any $k \geq K$. Therefore, $\phi(n_k + \phi(n_k))/\phi(n_k) \geq (\eta - \varepsilon)(1 + \eta - \varepsilon)n_k/\phi(n_k)$ for any $k \geq K$ and hence that

$$\limsup_{k \rightarrow \infty} \frac{\phi(n_k + \phi(n_k))}{\phi(n_k)} \geq \limsup_{k \rightarrow \infty} \frac{(\eta - \varepsilon)(1 + \eta - \varepsilon)n_k}{\phi(n_k)} = (1 + \eta - \varepsilon) \left(1 - \frac{\varepsilon}{\eta}\right).$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\limsup_{n \rightarrow \infty} \frac{\phi(n + \phi(n))}{\phi(n)} \geq 1 + \eta > 1,$$

which contradicts the condition $\phi \in \mathcal{H}$. \square

By the relation between $r_n(x)$ and $\ell_n(x)$, we state that

Lemma 6.8. *Let $\phi \in \mathcal{H}$. Then*

$$\left\{x \in [0, 1) : \limsup_{n \rightarrow \infty} \frac{\ell_n(x)}{\phi(n)} = 1\right\} \subseteq \left\{x \in [0, 1) : \limsup_{n \rightarrow \infty} \frac{r_n(x)}{\phi(n)} = 1\right\}.$$

Proof. On the one hand, for any $x \in [0, 1)$, in view of the definitions of $r_n(x)$ and $\ell_n(x)$, there exists $1 \leq k_n := k_n(x) < n$ such that $r_n(x) = \ell_{k_n}(x)$ and hence

$$\frac{r_n(x)}{\phi(n)} = \frac{\ell_{k_n}(x)}{\phi(n)} \leq \frac{\ell_{k_n}(x)}{\phi(k_n)}$$

since ϕ is positive and nondecreasing. Therefore,

$$\limsup_{n \rightarrow \infty} \frac{r_n(x)}{\phi(n)} \leq \limsup_{n \rightarrow \infty} \frac{\ell_{k_n}(x)}{\phi(k_n)} \leq \limsup_{n \rightarrow \infty} \frac{\ell_n(x)}{\phi(n)}. \quad (6.5)$$

We suppose that $\phi \in \mathcal{H}$. On the other hand, for any $x \in [0, 1)$, note that $r_{n+\ell_n(x)}(x) = \max_{1 \leq k \leq n} \ell_k(x)$, then $r_{n+\ell_n(x)}(x) \geq \ell_n(x)$ and hence

$$\limsup_{n \rightarrow \infty} \frac{r_n(x)}{\phi(n)} \geq \limsup_{n \rightarrow \infty} \frac{r_{n+\ell_n(x)}(x)}{\phi(n + \ell_n(x))} \geq \limsup_{n \rightarrow \infty} \left(\frac{\phi(n)}{\phi(n + \ell_n(x))} \cdot \frac{\ell_n(x)}{\phi(n)} \right). \quad (6.6)$$

Now let x be a real number satisfying $\limsup_{n \rightarrow \infty} \ell_n(x)/\phi(n) = 1$. Then there exists $N := N(x) > 0$ such that $0 \leq \ell_n(x) \leq 2\phi(n)$ for all $n \geq N$ and hence $\phi(n) \leq \phi(n + \ell_n(x)) \leq \phi(n + 2\phi(n))$ since ϕ is nondecreasing. By Lemma 6.6, we eventually deduce that

$$\lim_{n \rightarrow \infty} \frac{\phi(n)}{\phi(n + \ell_n(x))} = 1.$$

In view of (6.6), we obtain that

$$\limsup_{n \rightarrow \infty} \frac{r_n(x)}{\phi(n)} \geq \limsup_{n \rightarrow \infty} \left(\frac{\phi(n)}{\phi(n + \ell_n(x))} \cdot \frac{\ell_n(x)}{\phi(n)} \right) = \limsup_{n \rightarrow \infty} \frac{\ell_n(x)}{\phi(n)} = 1.$$

Combing this with (6.5), we complete the proof. \square

Theorem 6.9. *Let $\phi \in \mathcal{H}$. Then $\dim_{\mathcal{H}} G_\phi = 1$.*

Proof. In view of Lemma 6.8, we know that $F_\phi \subseteq G_\phi$ and hence it follows from Proposition 4.7 that

$$\dim_{\mathcal{H}} G_\phi \geq \frac{1}{1 + \liminf_{n \rightarrow \infty} \phi(n)/n}.$$

Let $\phi \in \mathcal{H}$. By Lemma 6.7, we have $\liminf_{n \rightarrow \infty} \phi(n)/n = 0$. Therefore, we get $\dim_{\mathcal{H}} G_\phi = 1$. \square

Corollary 6.10. *For any $\alpha \geq 0$,*

$$\dim_{\mathcal{H}} \left\{x \in [0, 1) : \limsup_{n \rightarrow \infty} \frac{r_n(x)}{\log_\beta n} = \alpha\right\} = 1.$$

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SCHOOL OF MATHEMATICS, SOUTH CHINA UNIVERSITY OF TECHNOLOGY, GUANGZHOU 510640, P.R. CHINA
E-mail address: f.lulu@mail.scut.edu.cn, wumin@scut.edu.cn and scbingli@scut.edu.cn